On Fairness and Randomness

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Abstract

We investigate the relation between the behavior of non-deterministic systems under fairness constraints, and the behavior of probabilistic systems. To this end, first a framework based on computable stopping strategies is developed that provides a common foundation for describing both fair and probabilistic behavior. On the basis of stopping strategies it is then shown that fair behavior corresponds in a precise sense to random behavior in the sense of Martin-Löf's definition of randomness.

We view probabilistic systems as concrete implementations of more abstract non-deterministic systems. Under this perspective the question is investigated what probabilistic properties are needed in such an implementation to guarantee (with probability one) certain required fairness properties in the behavior of the probabilistic system. Generalizing earlier concepts of ϵ-bounded transition probabilities, we introduce the notion of divergent probabilistic systems, which enables an exact characterization of the fairness properties of a probabilistic implementation. Looking beyond pure fairness properties, we also investigate what other qualitative system properties are guaranteed by probabilistic implementations of fair non-deterministic behavior. This leads to a completeness result which generalizes a well-known theorem by Pnueli and Zuck.

Key words: Fairness, Randomness, Probabilistic verification, Nondeterministic systems, Probabilistic systems.

1. Introduction

The concept of fairness has been introduced in the study of nondeterministic systems as a means to eliminate from the analysis of the system certain pathological behaviors. Fair behavior of a nondeterministic system is closely related to the (almost surely) expected behavior of a probabilistic system: if one replaces the nondeterministic transitions in a nondeterministic system by a probability distribution over the possible successor states, then one finds that the resulting probabilistic system will show a fair behavior with probability one [1].

To appear in Information and Computation

In fact, right from the beginning, some authors have viewed fairness conditions as a qualitative approximation to probabilistic behavior [2, 3]. De Alfaro **(year?)** has proposed to define fairness as a probabilistic concept altogether (probabilistic fairness).

Goal of this paper is to obtain a better understanding of the exact relationship between nondeterministic systems under fairness constraints, and probabilistic systems. Our first result is of a mostly conceptual, theoretical nature: we show (Section 2.3) that there exists an exact correspondence between system behaviors that satisfy certain fairness conditions, and system behaviors that are random in the classic sense of Martin-Löf [5, 6, 7]. More specifically, this result shows that for any given set of fairness constraints there exists a probabilistic system whose behavior will exhibit with probability one exactly the specified fairness properties. However, in this result the existence of such a probabilistic system is shown in a non-constructive manner, and therefore provides little guidance for the practical design of probabilistic systems with the desired fairness properties. The question of how a probabilistic system has to be designed in order to implement a given set of fairness conditions is investigated in Section 3. Our main result here establishes a necessary and sufficient condition for probabilistic implementations of given fairness conditions. Finally (Section 4), we investigate to what extent the fact that a given probabilistic system implements a certain set of fairness conditions is sufficient to ensure that the probabilistic system will only exhibit properties that are logically implied by these fairness conditions. Results that establish such a correspondence are known as completeness results [3, 1], because they show that logical deductive or model-checking techniques provide complete proof systems for deriving all probability one properties of a given probabilistic system. Our main result here is a generic completeness result that generalizes Pnueli and Zuck's **(year?)** result that α-fairness is complete for properties expressible in linear time temporal logic.

2. Fairness and Randomness

We begin by setting up the formal framework in which we study fairness and randomness. We require system models for nondeterministic and probabilistic systems (Section 2.1), and a way to describe fair, respectively random, system behaviors (Sections 2.2 and 2.3). It can then be shown that fair and random behavior coincides in a precise way (Theorem 2.13).

2.1. Nondeterministic and Probabilistic Systems

In previous works, concepts of fairness were usually defined with respect to particular types of (nondeterministic) system models that incorporated some features of an intended application domain, especially concurrent systems [8, 9, 10]. For the purpose of the present paper we can take a very abstract view, and use the most basic model of a nondeterministic transition system.

Definition 2.1. A nondeterministic transition system is a triple $G = (S, S_0, t)$, where S is a finite set of states, $S_0 \subseteq S$ is the set of starting states, and $t : S \to 2^S \setminus \emptyset$ is a transition relation.

The following notations are used for state sequences of G: ω denotes the set of natural numbers, $\sigma, \sigma', \ldots$ denote elements from S^ω, and $\boldsymbol{s}, \boldsymbol{s}', \ldots$ elements from S^*. The length of $\boldsymbol{s} \in S^*$ is denoted $|\, \boldsymbol{s} \,|$. The ith component of $\sigma \in S^\omega$ (resp. $\boldsymbol{s} \in S^*$ of length at least i) is denoted $\sigma[i]$ ($\boldsymbol{s}[i]$). We also use $\sigma[i, j]$ for the subsequence $\sigma[i], \sigma[i+1], \ldots, \sigma[j]$. For the prefix $\sigma[1, i]$ we simply write σ_i. $Run(G) := \{\sigma \mid \sigma[1] \in S_0, \sigma[i+1] \in t(\sigma[i])(i \in \omega)\}$ is the set of infinite runs of G, and $Run_{fin}(G) : \{\boldsymbol{s} \mid \boldsymbol{s}[1] \in S_0, \boldsymbol{s}[i+1] \in t(\boldsymbol{s}[i])(i <|\, \boldsymbol{s} \,|)\}$ is the set of finite runs of G.

A nondeterministic system G can be transformed into a probabilistic system by defining for each $s \in S$ a probability distribution on $t(s)$. When these probabilities are interpreted as transition probabilities, then we obtain a (stationary) Markov chain that defines a probability distribution on S^ω. We will call such a Markov chain an *implementation of* G, because it represents one way of turning the abstract nondeterministic model G into a concrete implementable system. G can also be implemented by systems other than Markov chains: for example, one can use time and history dependent transition probabilities, or, on the other hand, one can implement G using a fixed deterministic transition policy. In all cases, the implementation defines a probability distribution on $Run(G)$ (in the case of a deterministic implementation this is a degenerate distribution placing probability 1 on a single run $\sigma \in Run(G)$). Based on these considerations, we will in the following identify probabilistic systems (with set of states S) with probability distributions on S^ω.

The following standard definitions and notations pertain to probability distributions on S^ω: for $\boldsymbol{s} \in S^*$ with $|\, \boldsymbol{s} \,|= i$ we denote with $[\boldsymbol{s}]$ the *cylinder set* $\{\sigma \in S^\omega \mid \sigma_i = \boldsymbol{s}\}$. This notation is extended to $L \subseteq S^*$ via $[L] := \bigcup\{[\boldsymbol{s}] \mid \boldsymbol{s} \in L\}$. Let $\mathscr{A}$ denote the σ-algebra generated by the system $\{[\boldsymbol{s}] \mid \boldsymbol{s} \in S^*\}$. Throughout, we take $(S^\omega, \mathscr{A})$ to be the underlying probability space, and use $\mu, \mu', \ldots$ to denote probability measures on $(S^\omega, \mathscr{A})$. Usually, we speak loosely of μ as a probability measure on S^ω, when formally μ is meant to be a measure on $(S^\omega, \mathscr{A})$. To simplify notation, we express the probability of a set of sequences σ that satisfy some condition $c(\sigma)$ as $\mu(c(\sigma))$, rather than $\mu(\{\sigma \mid c(\sigma)\})$. All sets $\{\sigma \mid c(\sigma)\}$ that we shall encounter can easily be shown to be $\mathscr{A}$-measurable.

We can now formally state:

Definition 2.2. A probabilistic system over a finite set of states S is a probability distribution μ on $(S^\omega, \mathscr{A})$. A probabilistic system μ is called an *implementation* of a nondeterministic system G if $\mu(\boldsymbol{s}ss' \mid \boldsymbol{s}s) = 0$ for all $\boldsymbol{s} \in S^*$, $s \in S$ and $s' \notin t(s)$[1].

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[1] Obviously, only distributions satisfying certain computability conditions represent actually implementable systems, but we will not make this distinction here.

By Definitions 2.1 and 2.2 we have introduced extremely basic system models. As mentioned above, most previous investigations of fairness concepts were conducted on the basis of more elaborate system models. For the fundamental questions we are investigating, our simplified models are sufficient, and avoid an unnecessary burden of notation and detail. We will indicate in Section 5 how our results can be lifted to some more complex system models.

2.2. Nondeterministic Systems: Fairness

Numerous concepts of fairness have been proposed. These concepts broadly fall into two different categories. The first type of definition tries to directly capture the underlying intuition that in a fair computation "if a certain choice is possible sufficiently often, then it is sufficiently often taken" [11]. Approaches to formalize this intuition at various levels of generality include [9, 2, 3, 1, 12].

In all these definitions, the set of fair runs of a system is "large" in a topological sense. This has led to a second type of fairness definitions, in which fairness conditions are directly expressed by topological characterizations [13, 14, 15].

Since it is our goal to elucidate the close connection between the two fundamental conceptual notions of fairness and randomness, we follow the first approach to formalizing fairness. Like Baier and Kwiatkowska [1], we introduce a rather general framework for specifying fairness conditions of different strengths, which includes many previous more specific fairness concepts as special instances. Our definition is based on an adaptation of the concept of stopping times as used in the theory of stochastic processes. This approach is aimed at making transparent the connection with classical formalizations of randomness.

Definition 2.3. Let S be a finite set. A *stopping strategy for S* is a computable function

$$\tau : S^* \to \omega$$

such that $\tau(\boldsymbol{s}) \leq \tau(\boldsymbol{s}')$ when $\boldsymbol{s}$ is a prefix of $\boldsymbol{s}'$. The ith *stopping time* τ^i is defined for $\sigma \in S^\omega$ by

$$\tau^i(\sigma) := \min_k(\tau(\sigma_k) \geq i)) \ \in \omega \cup \{\infty\},$$

using the convention $\min \emptyset = \infty$, i.e. $\tau^i(\sigma) = \infty$ iff $\tau(\sigma_k) < i$ for all k. Similarly, for $\boldsymbol{s} \in S^*$:

$$\tau^i(\boldsymbol{s}) := \min_k(\tau(\boldsymbol{s}_k) \geq i)) \ \in \{1, \ldots, |\,\boldsymbol{s}\,|\} \cup \{\infty\}.$$

We denote with $\mathcal{T}(S)$ the set of all stopping strategies for S.

The underlying intuition behind the definition of a stopping strategy is that we read a sequence σ from left to right, and at selected positions $\sigma[k]$ increase an integer counter by some number. This process is equivalently modeled by τ, which gives the current counter value when the prefix $\boldsymbol{s}$ of σ has been read, or by the τ^i, which give the position of σ at which the counter value exceeds i. It is not essential that we are able to increase the counter by more than one at a single step, i.e. for all that follows we might as well have imposed the additional restriction $\tau(\boldsymbol{s}s) \leq \tau(\boldsymbol{s}) + 1$ for all $\boldsymbol{s} \in S^*, s \in S$ (or equivalently $\tau^i(\sigma) < \infty \Rightarrow \tau^i(\sigma) \neq \tau^{i+1}(\sigma)$ for all σ, i). The more general definition is merely a matter of convenience.

When $\tau^i(\sigma) < \infty$ then $\sigma[\tau^i(\sigma)] \in S$ is defined. In the following we simply write $\sigma[\tau^i]$ for $\sigma[\tau^i(\sigma)]$, and take equations of the form $\sigma[\tau^i] = s$ to stand for the conjunction " $\tau^i(\sigma) < \infty$ and $\sigma[\tau^i] = s$". Similarly, $\boldsymbol{s}[\tau^i] = s$ means "$\tau^i(\boldsymbol{s}) \leq \mid \boldsymbol{s} \mid$ and $\boldsymbol{s}[\tau^i(\boldsymbol{s})] = s$". In a similar vein, we shortly write $\{\tau^i < \infty\}$ for the set $\{\sigma \mid \tau^i(\sigma) < \infty\}$.

In connection with stopping times we abbreviate "for infinitely many i" with "i.o." (infinitely often).

Based on the notion of a stopping strategy, we define fairness as follows.

Definition 2.4. Let G be a nondeterministic transition system with states S, τ a stopping strategy for S, $\sigma \in S^\omega$. We say that σ is (τ, G)-fair, if for all $s \in S$ and $s' \in t(s)$:

$$\sigma[\tau^i] = s \text{ i.o.} \quad \Rightarrow \quad \sigma[\tau^i, \tau^i + 1] = ss' \text{ i.o.} \tag{1}$$

When T is a family of stopping strategies, we say that σ is (T, G)-fair if σ is (τ, G)-fair for all $\tau \in T$.

In the preceding definition the fairness condition (1) is required for all possible transitions from s. Sometimes one may want to impose such constraints only on a special subset of transitions. In our framework this corresponds to specifying for each state s a subset $t_{fair}(s) \subseteq t(s)$, and requiring (1) only for $s' \in t_{fair}(s)$. Our subsequent results can equally be developed in such a generalized setting. For the sake of conceptual and notational simplicity, however, we will not make such a distinction between $t(s)$ and $t_{fair}(s)$.

Example 2.5. (Strong fairness) Let $\tau_s(\boldsymbol{s}) :=\mid \boldsymbol{s} \mid$. Then τ_s is the strategy according to which one stops at every position of σ, The left hand side of (1) simply says that s appears infinitely often in σ. Condition (1) for all $s, s' \in t(s)$, thus, is the condition of strong fairness: every possible transition that is enabled infinitely often, is taken infinitely often.

Example 2.6. (α-fairness) Let ϕ be a formula in linear time temporal logic (LTL) over the set S as propositional variables. For $\boldsymbol{s} \in S^*$ then the decidable satisfaction relation $\boldsymbol{s} \models \phi$ is defined, and the stopping strategy $\tau_\phi(\boldsymbol{s}) :=\mid \{j \leq \mid \boldsymbol{s} \mid \mid \boldsymbol{s}_j \models \phi\} \mid$ expresses the rule that we stop whenever the prefix currently read satisfies ϕ. Let $T_\alpha := \{\tau_\phi \mid \phi \in \text{LTL over } S\}$. Then (T_α, G)-fairness is α-fairness as defined by Pnueli and Zuck **(year?)** (modulo the translation to our simpler system model).

Example 2.7. (Regular fairness) Let A be a finite automaton over input alphabet S, and $L(A) \subseteq S^*$ the language recognized by A. In analogy to the preceding example, define the stopping time τ_A by replacing the relation $\boldsymbol{s} \models \phi$ with $\boldsymbol{s} \in L(A)$. Let T_r be the set of such τ_A. The resulting concept of (T_r, G)-fairness is called regular fairness.

Example 2.8. (Computable fairness) The strongest possible fairness condition in our framework is $(\mathcal{T}(S), G)$-fairness, which we also call computable fairness. This concept of fairness was originally introduced in [16] (in a slightly different but equivalent way), and can be seen as the saturation of most previous fairness notions. The intuition behind computable fairness is unpredictability: if a sequence does not satisfy (1) for some τ, s, s', then an observer of the sequence can algorithmically make infinitely often a nontrivial and correct prediction on the behavior of the system by predicting that the next state will not be s' whenever τ stops at state s (more precisely, by starting to make these predictions after the finitely many occurrences of $\sigma[\tau^i, \tau^{i+1}] = ss'$ have passed). These predictions are nontrivial in the sense that they exclude a possible successor state of the current state. Computable fairness, now is the condition that a sequence is not algorithmically (partially) predictable in this sense.

Note that (T, G)-fairness of σ does not imply that σ is actually in $Run(G)$. Usually, however, we will only be interested in those (T, G)-fair sequences that are also runs of G, for which we introduce the notation

$$FairRun(T, G) := \{\sigma \in Run(G) \mid \sigma \text{ is } (T, G)\text{-fair}\}.$$

Depending on what kind of unknown or unspecified process is represented by the nondeterminism in the system, a more or less comprehensive fairness assumption will be appropriate: if, for instance, the nondeterminism models the policy of a scheduler that decides which of a number of concurrent processes is to make the next execution step, then a condition like strong fairness might be appropriate, as it simply excludes policies in which the execution of some processes will be stalled. A condition like computable fairness, on the other hand, would here be unreasonable, as it excludes perfectly sensible policies that schedule processes in a systematic, and hence predictable, round-robin fashion. When, in contrast, the nondeterministic system serves as an approximation for a probabilistic system, then the full condition of computable fairness becomes reasonable under suitable assumptions on the true underlying probabilistic system. The nature of these assumptions will be studied in Section 3.

As already mentioned, one important motivation for our specific formalization of fairness is to enable the comparison with the classical concept of randomness [5]. This comparison is facilitated by focusing on a special type of stopping strategies, which we now introduce.

The idea underlying the following definition is quite simple: given a stopping strategy τ, we can construct a new stopping strategy by picking some $k \in \omega, s, s' \in S$ and proceed as follows: read σ until the first k occurrences of subsequences of the form $\sigma[\tau^i, \tau^i + 1] = ss'$ have appeared, then stop whenever τ stops at the state s, but only until a further occurrence of $\sigma[\tau^i, \tau^i + 1] = ss'$ appears. The definition of the resulting new stopping strategy $\tau_{k,s,s'}(\boldsymbol{s})$ as a function on $\boldsymbol{s} \in S^*$ then is given as the number of times j that one has stopped when reading $\boldsymbol{s}$. The precise definition of the resulting stopping strategy $\tau_{k,s,s'}$ is as follows.

Definition 2.9. Let τ be a stopping strategy, $k \in \omega,\ s, s' \in S$. The (k, s, s')-*transform* of τ is the stopping strategy $\tau_{k,s,s'}$ defined as follows: $\tau_{k,s,s'}(\boldsymbol{s})$ is the largest $j \in \omega$ such that there exist $i_0, i_1 \in \omega$ with

(a) $|\{i \leq i_0 \mid \boldsymbol{s}[\tau^i, \tau^i + 1] = ss'\}| = k$

(b) $|\{i_0 < i \leq i_1 \mid \boldsymbol{s}[\tau^i] = s\}| = j$

(c) $\forall i : i_0 < i \leq i_1 - 1 : \boldsymbol{s}[\tau^i, \tau^i + 1] \neq ss'$

A set $T \subseteq \mathcal{T}(S)$ is said to be *closed under $\tau_{k,s,s'}$-transforms*, if $\tau \in T$ implies $\tau_{k,s,s'} \in T$ for all k, s, s'.

Stopping strategies of the form $\tau_{k,s,s'}$ have some special properties that we will exploit in the following sections. Since $\tau_{k,s,s'}$ only stops at state s, one obtains as an immediate consequence of our definitions:

$$\sigma[\tau^i_{k,s,s'} = s] \Leftrightarrow \tau^i_{k,s,s'}(\sigma) < \infty. \tag{2}$$

Another immediate consequence from the definitions is that for all σ, G, and all k, s, s' with $s' \in t(s)$:

$$\sigma \text{ is } (\tau_{k,s,s'}, G)\text{-fair} \Leftrightarrow \sigma \notin \cap_{i=1}^{\infty}\{\tau^i_{k,s,s'} < \infty\}. \tag{3}$$

The following lemma, which again follows directly from the definitions, shows that k, s, s'-transforms are in a sense sufficient.

Lemma 2.10. *Let G be a nondeterministic transition system with states S, $\tau \in \mathcal{T}(S)$, $\sigma \in S^\omega$. The following are equivalent:*

(i) *σ is (τ, G)-fair*

(ii) *σ is $(\tau_{k,s,s'}, G)$-fair for all $k \in \omega,\ s \in S,\ s' \in t(s)$.*

Lemma 2.10 permits us to restrict attention to stopping strategies of the form $\tau_{k,s,s'}$ whenever we are dealing with a family $T \subseteq \mathcal{T}(S)$ that is closed under k, s, s'-transforms. Most natural families have this property: for example, T_α, T_r and $\mathcal{T}(S)$ (cf. Examples 2.6-2.8) can be shown to be closed under k, s, s'-transforms.

2.3. Probabilistic systems: Randomness and Fairness

The problem of how to distinguish random from non-random sequences has a long history. Von Mises [17] was the first to propose a formal definition of randomness, which, however, was not fully consistent. Capturing the underlying intuition of von Mises's approach, Martin-Löf's **(year?)** provided a now classical definition of randomness. For related, alternative, definitions see e.g. [6, 18], and [19] for an overview.

The basic idea behind this definition is that a sequence σ is random (with regard to a given probability distribution μ) if σ does not belong to an "exceptional" set C with $\mu(C) = 0$. This idea cannot be directly used as a formal definition, because typically one has $\mu(\{\sigma\}) = 0$ for all $\sigma \in S^\omega$, so that no random sequences would exist. For this reason, Martin-Löf restricts the sets C under consideration to those for which membership $\sigma \in C$ is testable by an effectively computable sequential statistical test. Such tests can be defined in several equivalent ways. In the following we give a definition in terms of stopping strategies in the sense of Definition 2.3.

Definition 2.11. Let μ be a probability measure on S^ω. A stopping strategy τ is called a *test for* μ if $\mu(\cap_{i=1}^\infty \{\tau^i < \infty\}) = 0$. A sequence $\sigma \in S^\omega$ is called (τ, μ)-random, if τ is not a test for μ, or $\sigma \notin \cap_{i=1}^\infty \{\tau^i < \infty\}$. If T is a family of stopping strategies, then σ is called (T, μ)-random if σ is (τ, μ)-random for all $\tau \in T$.

Example 2.12. Let $S = \{0, 1\}$, and μ be the uniform distribution on S^ω (i.e. $\mu([\boldsymbol{s}]) = 1/2^{|\boldsymbol{s}|}$ for all $\boldsymbol{s} \in S^*$). Let $\epsilon > 0$, and

$$\tau_\epsilon(\boldsymbol{s}) :=| \{ i \leq |\, \boldsymbol{s} \,| \mid \frac{1}{i} \sum_{j=1}^{i} \boldsymbol{s}[j] > \frac{1}{2} + \epsilon \} \mid .$$

Thus, $\tau_\epsilon(\boldsymbol{s})$ counts how often the mean value of the states in $\boldsymbol{s}$ exceeds $1/2 + \epsilon$. By the strong law of large numbers we have

$$\mu(lim_{i \to \infty} \frac{1}{i} \sum_{j=1}^{i} \sigma[j] = \frac{1}{2}) = 1,$$

which means that

$$\mu(\{i \in \omega \mid \frac{1}{i} \sum_{j=1}^{i} \sigma[j] > \frac{1}{2} + \epsilon\} \text{ is finite }) = 1.$$

Since finiteness of $\{i \in \omega \mid \frac{1}{i} \sum_{j=1}^{i} \sigma[j] > \frac{1}{2} + \epsilon\}$ is equivalent to $\sigma \notin \cap_{i=1}^\infty \{\tau_\epsilon^i < \infty\}$, one obtains that τ_ϵ is a test for μ (for any $\epsilon > 0$).

The notion of $(\mathcal{T}(S), \mu)$-randomness as given by Definition 2.11 is equivalent to Martin-Löf's definition of randomness, modulo the following two changes: in order to obtain Martin-Löf's original concept, one would have to restrict Definition 2.11 to computable μ, and replace the condition $\mu(\cap_{i=1}^\infty \{\tau^i < \infty\}) = 0$ (which is equivalent to $\mu(\tau^i < \infty) \to 0$ for $i \to \infty$) with the stronger condition $\mu(\tau^i < \infty) < 1/2^i$ $(i \in \omega)$. Both these modifications are instrumental for the construction of a universal test of randomness, one of the main goals of Martin-Löf's work. From the point of view of providing a natural definition of randomness, however, the more general notion of a test as given in Definition 2.11 seems to be rather more appropriate than the one used by Martin-Löf; it has previously also been adopted by Gaifman and Snir **(year?)**.

On the basis of the common foundation in terms of stopping strategies for fairness and randomness, we can now establish an exact correspondence between these two notions.

Theorem 2.13. *Let G be a nondeterministic transition system with states S; let $T \subseteq \mathcal{T}(S)$. There exists a probability measure $\tilde{\mu}$ such that for all $\sigma \in S^\omega$: σ is (T,G)-fair iff σ is $(\mathcal{T}(S),\tilde{\mu})$-random.*

Proof: Define $\mathcal{M} := \{\mu \mid \mu((T,G)\text{-fair}) = 1\}$. A stopping strategy τ is called a test for $\mathcal{M}$, if τ is a test for all $\mu \in \mathcal{M}$. Let $\tau_1, \tau_2, \ldots$ be the (finite or countably infinite) set of stopping strategies in T that are not tests for $\mathcal{M}$. For each $j \in \omega$ let $\mu_j \in \mathcal{M}$ be such that $\mu_j(\cap_{i=1}^{\infty}\{\tau_j^i < \infty\}) > 0$, and define $\tilde{\mu} = \sum_{j=1}^{\infty} 1/2^j \mu_j$. The constructed $\tilde{\mu}$ has the following property: a stopping strategy τ is a test for $\mathcal{M}$ iff τ is a test for $\tilde{\mu}$.

Now consider $\sigma \in S^\omega$, and assume that σ is $(\mathcal{T}(S),\tilde{\mu})$-random. To show that σ is (T,G)-fair it is sufficient to show that $\sigma \notin \cap_{i=1}^{\infty}\{\tau_{k,s,s'}^i < \infty\}$ for any k,s,s'-transform of any $\tau \in T$ (Lemma 2.10 and (3)). Let τ, k, s, s' be given. For any $\mu \in \mathcal{M}$, by definition of $\mathcal{M}$ and Lemma 2.10, then $\mu((\tau_{k,s,s'},G)\text{-fair}) = 1$, and hence $\mu(\cap_{i=1}^{\infty}\{\tau_{k,s,s'}^i < \infty\}) = 0$. Then also $\tilde{\mu}(\cap_{i=1}^{\infty}\{\tau_{k,s,s'}^i < \infty\}) = 0$, i.e. $\tau_{k,s,s'}$ is a test for $\tilde{\mu}$. $(\mathcal{T}(S),\tilde{\mu})$-randomness of σ now implies $\sigma \notin \cap_{i=1}^{\infty}\{\tau_{k,s,s'}^i < \infty\}$, as required.

For the converse direction, assume that σ is (T,G)-fair. Let ϵ_σ denote the point mass on σ, i.e. the probability distribution that assigns probability 1 to every set that contains σ. Then $\epsilon_\sigma \in \mathcal{M}$. Let $\tau \in \mathcal{T}(S)$ be a test for $\tilde{\mu}$. Then τ also is a test for $\mathcal{M}$, and hence for ϵ_σ, so that $\epsilon_\sigma(\cap_{i=1}^{\infty}\{\tau^i < \infty\}) = 0$. By definition of ϵ_σ, now, $\sigma \notin \cap_{i=1}^{\infty}\{\tau^i < \infty\}$, i.e. σ is $(\mathcal{T}(S),\tilde{\mu})$-random.

□

It is important to note that the equivalence in Theorem 2.13 is between (T,G)-fairness and $(\mathcal{T}(S),\mu)$-randomness, and not between (T,G)-fairness and (T,μ)-randomness. Thus, fairness is shown to be equivalent to randomness in the full sense of Martin-Löf **(year?)**, and not only equivalent to a weakened form of randomness, determined by only the subset T of stopping strategies. The class T of stopping strategies defining the fairness concept under consideration only influences the construction of the probability measure $\tilde{\mu}$.

3. Probabilistic Fairness Guarantees

Probabilistic implementations of a nondeterministic system will typically possess quite general fairness guarantees. This has especially been noted for Markov chain implementations [1]. Theorem 2.13 establishes an exact correspondence between fairness and probabilistic behavior. In particular, the theorem shows that for every fairness concept T there exists a probabilistic implementation $\tilde{\mu}$ that guarantees (T,G)-fairness with probability one. However, $\tilde{\mu}$ is obtained in a non-constructive manner in the proof of Theorem 2.13, and is not even guaranteed to be computable. In this section we take a more constructive approach to the question of how to obtain probabilistic implementations of certain fairness properties (or, conversely, what fairness properties will be guaranteed by a given implementation). Our interest in this section is mostly with probabilistic systems that are not Markov chains, and therefore may not readily be seen to possess the required fairness properties. The main probabilistic concept needed in the investigation of such systems is given in the following definition.

Definition 3.1. Let G be a nondeterministic transition system, τ a stopping strategy, and μ a probability measure on S^ω. We say that μ is (τ, G)-*divergent*, if for all $s \in S$ and $s' \in t(s)$:

$$\mu(\sigma[\tau^i] = s \text{ i.o.}) > 0 \qquad \Rightarrow \sum_{\substack{i=1 \\ \mu(\sigma[\tau^i]=s)>0}}^{\infty} \mu(\sigma[\tau^i + 1] = s' \mid \sigma[\tau^i] = s) \ = \infty \tag{4}$$

(note that $\mu(\sigma[\tau^i] = s \text{ i.o.}) > 0$ implies that $\mu(\sigma[\tau^i] = s) > 0$ for infinitely many i, but not vice versa). If T is a family of stopping strategies, then μ is called (T, G)-divergent if μ is (τ, G)-divergent for all $\tau \in T$.

Lemma 3.2. *Let G be a nondeterministic transition system, $s, s' \in S$ with $s' \in t(s)$. Let $\tau \in \mathcal{T}(S)$ and $k \in \omega$. Then*

$$\mu((\tau_{k,s,s'}, G)\textit{-fair}) = 1 \Leftrightarrow \mu \textit{ is } (\tau_{k,s,s'}, G)\textit{-divergent.} \tag{5}$$

Proof: For $\tau_{k,s,s'}$ we have by (2) and (3)

$$\sigma \text{ is } (\tau_{k,s,s'}, G)\text{-fair} \Leftrightarrow \text{not } \sigma[\tau^i_{k,s,s'}] = s \text{ i.o.}$$

The left to right direction of the theorem directly follows, because $\mu((\tau_{k,s,s'}, G)\text{-fair}) = 1$ makes (4) vacuously true with $\mu(\sigma[\tau^i_{k,s,s'}] = \tilde{s} \text{ i.o.}) = 0$ for all $\tilde{s} \in S$.

For the converse direction, assume that (4) holds for $\tau_{k,s,s'}$. We need to show that $\mu(\sigma[\tau^i_{k,s,s'}] = s \text{ i.o.}) = 0$. Assume otherwise, i.e. $\mu(\sigma[\tau^i_{k,s,s'}] = s \text{ i.o.}) = r > 0$. This implies $\mu(\sigma[\tau^i_{k,s,s'}] = s) \geq r$ for all i. With $\{\tau^{i+1}_{k,s,s'} < \infty\} \subseteq \{\tau^i_{k,s,s'} < \infty\} \cap \{\tau^{i+1}_{k,s,s'} = s'\}^c$, and using (2), it follows that

$$\begin{aligned} &\mu(\sigma[\tau^i_{k,s,s'} + 1] = s' \mid \sigma[\tau^i_{k,s,s'}] = s) \leq \mu(\tau^{i+1}_{k,s,s'} = \infty \mid \tau^i_{k,s,s'} < \infty) \\ &= \frac{\mu(\tau^i_{k,s,s'} < \infty, \tau^{i+1}_{k,s,s'} = \infty)}{\mu(\tau^i_{k,s,s'} < \infty)} \leq \frac{\mu(\tau^i_{k,s,s'} < \infty, \tau^{i+1}_{k,s,s'} = \infty)}{r}. \end{aligned}$$

Summing over all i, we obtain ∞ on the left hand side and $(1 - r)/r$ on the right hand side, a contradiction. □

The following example shows that (5) does not hold for arbitrary stopping strategies τ.

Example 3.3. Let G be a nondeterministic coin-tossing model: $G = (\{\mathsf{H}, \mathsf{T}\}, \{\mathsf{H}, \mathsf{T}\}, t)$ with $t(\mathsf{H}) = t(\mathsf{T}) = \{\mathsf{H}, \mathsf{T}\}$. Consider the following probabilistic implementations of G: μ_1, the point mass on H^ω (modeling a bogus coin with heads on both sides), μ_2, the uniform distribution on $\{\mathsf{H}, \mathsf{T}\}^\omega$ (modeling a fair coin), and

$\mu := 1/2(\mu_1 + \mu_2)$, a mixture of μ_1 and μ_2. Define $\tau(\boldsymbol{s}) :=\mid\{i \leq \mid \boldsymbol{s} \mid \mid \boldsymbol{s}[i] = \mathsf{H}\}\mid$, i.e. τ stops at every occurrence of H. Then μ is (τ, G)-divergent, because for all i: $\mu(\sigma[\tau^i] = \mathsf{H}) = 1$,

$$\mu(\sigma[\tau^{i+1}] = \mathsf{T} \mid \sigma[\tau^i] = \mathsf{H}) = \\ 1/2(\mu_1(\sigma[\tau^{i+1}] = \mathsf{T}) + \mu_2(\sigma[\tau^{i+1}] = \mathsf{T})) = 1/2(0 + 1/2),$$

and $\mu(\sigma[\tau^{i+1}] = \mathsf{H} \mid \sigma[\tau^i] = \mathsf{H}) = 1/2(1 + 1/2)$.

On the other hand, we have that

$$\mu((\tau, G)\text{-fair}) = 1/2(\mu_1((\tau, G)\text{-fair}) + \mu_2((\tau, G)\text{-fair})) = 1/2(0 + 1) = 1/2.$$

Hence (5) does not hold for $T = \{\tau\}$.

Even though (5) does not hold "pointwise" for arbitrary τ, we show by the following theorem that this equivalence can be extended to most relevant classes of stopping strategies.

Theorem 3.4. *Let $T \subseteq \mathcal{T}(S)$ be closed under k, s, s'-transforms. Then*

$$\mu((T, G)\text{-}\mathit{fair}) = 1 \quad \mathit{iff} \quad \mu \text{ is } (T, G)\text{-}\mathit{divergent}.$$

Proof: "$\Leftarrow$": Since T is countable, we have that $\mu((T, G)\text{-fair}) = 1$ is equivalent to $\mu((\tau, G)\text{-fair}) = 1$ for all $\tau \in T$. By Lemma 2.10 the latter is equivalent $\mu((\tau_{k,s,s'}, G)\text{-fair}) = 1$ for all k, s, s'-transforms of $\tau \in T$. Using Lemma 3.2 and the closure of T under k, s, s'-transforms, this is implied by the (T, G)-divergence of μ.

"$\Rightarrow$": Assume that μ is not (T, G)-divergent, i.e. there exists $\tau \in T$ with $\mu(\sigma[\tau^i] = s \text{ i.o.}) = r > 0$ and $\sum \mu(\sigma[\tau^i + 1] = s' \mid \sigma[\tau^i] = s) < \infty$. As $\mu(\sigma[\tau^i, \tau^i + 1] = ss') \leq \mu(\sigma[\tau^i + 1] = s' \mid \sigma[\tau^i] = s)$, we have by the first Borel-Cantelli lemma (e.g. [20, Theorem 4.3]) that $\mu(\sigma[\tau^i, \tau^i + 1] = ss' \text{ i.o.}) = 0$. Thus $1 - r \geq \mu((\tau, G)\text{-fair}) \geq \mu((T, G)\text{-fair})$ □

When μ is an implementation of G with $\mu((T, G)\text{-fair}) = 1$, then we also say that μ is a (T, G)-fair implementation of G.

Theorem 3.4 provides an exact characterization of (T, G)-fair implementations of G, provided T has the required closure property. For T not closed under k, s, s'-transforms, still the left-to-right implication of Theorem 3.4 holds, i.e. one obtains a necessary probabilistic condition for μ being a (T, G)-fair implementations of G. As Example 3.3 illustrates, it may then not be a sufficient one. This limitation of Theorem 3.4 does not appear to be a very serious one, as closure under k, s, s'-transforms is a rather natural property for classes of stopping strategies.

Example 3.5. (Markov chains) Let G be a nondeterministic transition system. Define a Markov chain on G by assigning to every $s \in S_0$ a starting probability $p_0(s)$, and to every pair s, s' with $s' \in t(s)$ a transition probability $p(s, s') > 0$, such that $\sum_{s' \in t(s)} p(s,s') = 1$. Let μ be the probability distribution defined by the Markov chain on S^ω. Then μ is $(\mathcal{T}(S), G)$-divergent. A slight generalization is obtained by allowing non-stationary Markov chains with ϵ-bounded transition probabilities: here the transition probabilities are defined separately in the form $p_i(s,s')$ for each time point $i \in \omega$. If the $p_i(s,s')$ satisfy a global lower bound $p_i(s,s') \geq \epsilon > 0$, then the induced distribution μ is $(\mathcal{T}(S), G)$-divergent, and hence $\mu((\tau, G)\text{-fair}) = 1$ for all τ. This is essentially Theorem 1 of Baier and Kwiatkowska **(year?)** (with the difference that Baier and Kwiatkowska considered ϵ-bounds for transitions in a stationary infinite state Markov chain, whereas here we consider non-stationary finite state Markov chains).

Finally, one can even omit the condition that transition probabilities are Markovian, and define for each $\boldsymbol{s} \in Run_{fin}(G)$ ending with s, and $s' \in t(s)$ transition probabilities $p(\boldsymbol{s}, s')$. If all these transition probabilities satisfy a global lower bound $p(\boldsymbol{s}, s') > \epsilon$, then the induced distribution μ still is $(\mathcal{T}(S), G)$-divergent.

Example 3.6. Let G, μ and τ be as in Example 3.3. Let T be the closure of $\{\tau\}$ under k, s, s'-transforms. Then μ is not (T, G)-divergent. To see this, consider the stopping strategy $\tilde{\tau} := \tau_{0,\mathsf{H},\mathsf{T}}$. This strategy consists of stopping at every H until the first T appears.

We then have $\sigma[\tilde{\tau}^i] = \mathsf{H}$ infinitely often iff $\sigma = \mathsf{H}^\omega$, and hence $\mu(\sigma[\tilde{\tau}^i] = \mathsf{H}\text{ i.o.}) = 1/2$. Also, for every $i \in \omega$: $\mu(\sigma[\tilde{\tau}^i] = \mathsf{H}) > 0$ and

$$\begin{aligned}
&\mu(\sigma[\tilde{\tau}^{i+1}] = \mathsf{T} \mid \sigma[\tilde{\tau}^i] = \mathsf{H}) = \mu(\sigma[\tilde{\tau}^{i+1}] = \mathsf{T} \mid \tilde{\tau}^i < \infty) = \\
&= \frac{\mu_1(\sigma[\tilde{\tau}^{i+1}] = \mathsf{T}, \tilde{\tau}^i < \infty) + \mu_2(\sigma[\tilde{\tau}^{i+1}] = \mathsf{T}, \tilde{\tau}^i < \infty)}{\mu_1(\tilde{\tau}^i < \infty) + \mu_2(\tilde{\tau}^i < \infty)} \\
&= \frac{0 + (1/2)^{i+1}}{1 + (1/2)^i} \leq (1/2)^{i+1}.
\end{aligned}$$

Thus, the defining condition (4) of (τ, G)-divergence does not hold for $\tilde{\tau}, s = \mathsf{H}$, and $s' = \mathsf{T}$.

The following more elaborate example gives an application of Theorem 3.4 in a situation where no global ϵ-bound on transition probabilities can be given.

Example 3.7. Consider a salesclerk who has to serve customers arriving at two different counters c, c'. The salesclerk needs a constant time unit to serve one customer. While serving a customer, new customers arrive randomly and independently at the two counters. Assume that the number of customers arriving at counters c, c' during the ith time unit is given by random variables n_i, n'_i, whose distributions ν, ν' do not depend on i. Let $0 < \lambda, \lambda' < \infty$ be the expected values of ν, respectively ν'.

Assume that the salesclerk uses a randomized strategy to decide which counter to serve in the next time unit: if p, p' denote, respectively, the number of people currently waiting at counters c, c', then the clerk will next serve counter c with probability $p/(p+p')$. One can easily see that in an idealized, deterministic version of this scenario, this will be the strategy that ensures that customers arriving in the same time unit will have to wait approximately equally long. For our probabilistic model, we would like to prove a much weaker fairness property, namely, that with probability one all customers arriving at either of the two counters will be served eventually. The intuitive reason why this is the case is the following: disregarding for a moment that customers also leave the queues after being served, we obtain from the strong law of large numbers that with probability one the number of people waiting at the two counters will show an approximately linear growth with linear coefficients λ, λ'. This means that the probability that c will be served next is approximately $\lambda/(\lambda+\lambda')$ at every point in time, and thus, with probability one, c will be chosen infinitely often. Taking the departure of customers into account, we obtain as a lower bound for the number of customers waiting at c a linear growth with coefficient $\max\{0, \lambda-1\}$ (note that $\lambda-1$ is the growth rate at counter c if c is always chosen for service). Assuming that $\lambda > 1$, we still obtain a nonzero constant lower bound $(\lambda-1)/(\lambda-1+\lambda')$ for the selection probability of c. In the following, we will use Theorem 3.4 to turn this intuitive reasoning for the case $\lambda > 1, \lambda' > 1$ into a rigorous proof.

Our scenario is naturally modeled as a Markov chain over state space $S^+ = \{c, c'\} \times \omega \times \omega$, where (c, k, k') represents the state that in the current time unit counter c is being served, and at the end of the current time unit k and k' customers are waiting at counters c, respectively c'. The transition probability from e.g. (c, k, k') to (c', l, l') then is given by $k'/(k+k')\nu(l-k)\nu'(l'-k'+1)$. Let μ denote the probability distribution on $(S^+)^\omega$ induced by this Markov chain. Since we are here dealing with an infinite state Markov chain whose transition probabilities are not ϵ-bounded in the sense of Example 3.5, none of our results so far directly apply. However, for the analysis of our fairness condition we are only interested in the question whether a state sequence generated by the Markov chain will contain both infinitely many states of the form $(c, \cdot, \cdot)$ and $(c', \cdot, \cdot)$. Relevant to our question, thus, is the marginal distribution induced by μ on sequences over $S = \{c, c'\}$. We denote this distribution by $\bar{\mu}$. Formally, for any finite sequence $(c_1, \ldots, c_n) \in S^*$:

$$\bar{\mu}(c_1, \ldots, c_n) = \sum_{k_1, k'_1, \ldots, k_n, k'_n \in \omega} \mu((c_1, k_1, k'_1), \ldots, (c_n, k_n, k'_n)).$$

The precise question, now, is whether $\bar{\mu}((T_{strong}, G)\text{-fair}) = 1$, where G is the transition system over S with $t(c) = t(c') = \{c, c'\}$. We will actually show the much stronger result that $\bar{\mu}((\mathcal{T}(S), G)\text{-fair}) = 1$. Since $\mathcal{T}(S)$ is closed under k, s, s'-transforms, we obtain this result by showing that (4) holds for k, s, s'-transforms in $\mathcal{T}(S)$.

Let $N_i := \sum_{j=1}^{i} n_j$, $N'_i := \sum_{j=1}^{i} n'_j$ be the random variables representing the total number of customers that have arrived at the two counters up to and including time unit i. At time step i the underlying Markov chain then is in state $(\cdot, k, k')$, where $k\,(k')$ is $N_i\,(N'_i)$ minus the number of times $c\,(c')$ was served. By the strong law of large numbers we have that for every $\epsilon > 0$: $\mu(lim_i N_i/i \in [\lambda-\epsilon, \lambda+\epsilon]) = 1$, and similarly for N'_i (note that the N_i can be retrieved from the state sequence in S^+, so that μ really specifies these probabilities). Choose $\epsilon > 0$ with $\lambda - 1 - \epsilon > 0$ and $\lambda' - 1 - \epsilon > 0$.

Now let τ be a k, s, s'-transforms in $\mathcal{T}(S)$. For concreteness' sake, assume that τ has the form $\tau_{k,c,c'}$. To show (4) assume that $\mu(\sigma[\tau^i] = c \text{ i.o.}) = r > 0$ (otherwise we are done). This implies $\mu(\sigma[\tau^i] = c) \geq r$ for all i. Now:

$$\begin{aligned}\bar{\mu}(\sigma[\tau^i + 1] = c' \mid \sigma[\tau^i] = c) \geq \\ \mu(\sigma[\tau^i + 1] = c' \mid \sigma[\tau^i] = c, \mid N_{\tau^i}/\tau^i - \lambda \mid < \epsilon, \mid N'_{\tau^i}/\tau^i - \lambda' \mid < \epsilon) \\ \cdot \mu(\mid N_{\tau^i}/\tau^i - \lambda \mid < \epsilon, \mid N'_{\tau^i}/\tau^i - \lambda' \mid < \epsilon \mid \sigma[\tau^i] = c) \end{aligned} \quad (6)$$

First consider the first factor on the right hand side of (6). $\mid N_{\tau^i}/\tau^i - \lambda \mid < \epsilon$ and $\mid N'_{\tau^i}/\tau^i - \lambda' \mid < \epsilon$ imply that the Markov chain at time τ^i is in a state (c, k, k'), where $k' \geq N'_{\tau^i} - \tau^i \geq (\lambda' - 1 - \epsilon)\tau^i$, and $k \leq N_{\tau^i} - 1 \leq (\lambda + \epsilon)\tau^i$. Then $k'/(k + k') \geq (\lambda' - 1 - \epsilon)/(\lambda' - 1 + \lambda)$. Since $\lambda' - 1 - \epsilon > 0$ this gives a strictly positive lower bound for the first factor. Turning to the second factor, we obtain from $\tau^i \geq i$ that

$$\{\mid N_{\tau^i}/\tau^i - \lambda \mid < \epsilon, \mid N'_{\tau^i}/\tau^i - \lambda' \mid < \epsilon\} \supseteq \cap_{k \geq i}\{\mid N_k/k - \lambda \mid < \epsilon, \mid N'_k/k - \lambda' \mid < \epsilon\}$$

For $i \to \infty$ the probability of the intersection on the right converges to 1, and, hence, so does the probability of the set on the left. Because of the uniform lower bound r for $\mu(\sigma[\tau^i] = c)$, we obtain that the second factor in (6) converges to 1 as $i \to \infty$. Combining the result for the two factors, we obtain that the right hand side of (6) is bounded from below by a strictly positive constant as $i \to \infty$, and hence gives ∞ when summed over all i.

4. Completeness

In the previous sections we have investigated under what conditions a probabilistic system will satisfy certain fairness conditions. However, fairness conditions are usually not the final goal, but only a means to establish the properties of ultimate interest, e.g. program termination [9]. Termination can be represented in our simplified system model by a designated goal state $g \in S$, i.e. a run $\sigma \in S^\omega$ is terminating iff $\sigma[i] = g$ for some i. Thus, we can define the set of all terminating runs of system G:

$$C_{term} := \{\sigma \in \mathit{Run}(G) \mid \exists i : \sigma[i] = g\}.$$

The question of whether a certain fairness assumption $T \subseteq \mathcal{T}(S)$ guarantees termination for G then is the question of whether the inclusion

$$\mathit{FairRun}(T, G) \subseteq C_{term} \quad (7)$$

is valid. When μ is a probabilistic implementation of G with

$$\mu(\mathit{FairRun}(T, G)) = 1, \quad (8)$$

then (7) implies

$$\mu(C_{term}) = 1. \tag{9}$$

Conversely, (8) and (9) together will, in general, not imply (7), because (9) may hold only because of some special features of the probabilistic system μ. For some other (T, G)-fair implementation of G then perhaps $\tilde{\mu}(C_{term}) \neq 1$, which, in particular, precludes $\mathit{FairRun}(T, G) \subseteq C_{term}$.

A problem of considerable interest, now, is to identify certain classes of properties C, fairness conditions T, and (T, G)-fair implementations μ of G, such that, in fact

$$\mu(C) = 1 \quad \Leftrightarrow \quad \mathit{FairRun}(T, G) \subseteq C. \tag{10}$$

When this equivalence holds, then logic-based and probabilistic methods for verification can be used interchangeably for property C: the inclusion on the right-hand side of (10) represents a verification problem for the nondeterministic system G, which would usually be solved by theorem proving or model checking techniques. When μ is a (T, G)-fair implementation of G for which (10) holds, then one can alternatively use probabilistic methods to show that $\mu(C) = 1$. Conversely, when we are given a probabilistic system μ, then the left-hand side of (10) expresses a probabilistic verification problem for μ. When μ is known to be a (T, G)-fair implementation of G, such that (10) holds, then logic-based methods can be used to solve the probabilistic verification problem. Because of this latter perspective, an equivalence of the form (10) is called a completeness result [3, 1] (logic-based methods are complete for probabilistic verification).

Our goal is to identify classes of probabilistic systems $\mathcal{M}$, classes of properties $\mathcal{C} \subseteq 2^{S^\omega}$, and fairness conditions $T \subseteq \mathcal{T}(S)$, such that (10) holds for all $\mu \in \mathcal{M}$, $C \in \mathcal{C}$, and T. The classical result of this type is Pnueli and Zuck's **(year?)** theorem that (10) holds when $\mathcal{C}$ is the class of all LTL-definable properties, $\mathcal{M}$ is the class of Markov chains on G (cf. Example 3.5), and T is the condition of α-fairness (cf. Example 2.6) (again adapting Pnueli and Zuck's result to our simpler system models).

Observe that when (10) can be shown to hold for all μ from a certain class $\mathcal{M}$, then this also entails an important *robustness* property of C: since μ does not appear on the right-hand side of (10), one obtains that $\mu(C) = 1$ holds either for all $\mu \in \mathcal{M}$, or for no $\mu \in \mathcal{M}$.

Let us denote by

$$\mathcal{M}_{FR(T,G)} := \{\mu \mid \mu(\mathit{FairRun}(T, G))) = 1\}$$

the set of (T, G)-fair implementations of G. When $\mu \in \mathcal{M}_{FR(T,G)}$, then the right-to-left implication of (10) is trivial. Furthermore, when $\mathit{FairRun}(T, G) \in \mathcal{C}$, then (10) can only hold when $\mu \in \mathcal{M}_{FR(T,G)}$. In the previous sections we obtained tools for deciding whether a given μ is a member of $\mathcal{M}_{FR(T,G)}$. In the present section, therefore, we will focus on classes $\mathcal{M} \subseteq \mathcal{M}_{FR(T,G)}$, and the left-to-right implication of (10), the converse then being trivial. Furthermore, our main result will refer to finite automata both for the definition of the class $\mathcal{C}$, and for the fairness condition T. In the following subsection we summarize some well-known facts about infinitary languages $\mathcal{C}$ defined by finite automata.

4.1. Infinitary Languages

We write $A = (S, Z, z_0, r, Z_{acc})$ for a deterministic finite automaton with input alphabet S, set of states Z , initial state z_0, transition function r, and set of accepting states $Z_{acc} \subseteq Z$. We assume that r is deterministic, but possibly incomplete, i.e. it is a partial function from $Z \times S$ to Z. We refer to an automaton with input alphabet S as an S-automaton.

An infinite input string $\sigma \in S^\omega$ induces a finite or infinite state sequence $\zeta \in Z^* \cup Z^\omega$, depending on whether σ leads to a transition not defined by r. We denote with $\mathit{inf}_A(\sigma) \subseteq Z$ the (possibly empty) set of states that occur infinitely often in ζ.

A finite automaton is turned into a Muller automaton by replacing Z_{acc} with a set $\mathcal{Z} \subseteq 2^Z$. A sequence $\sigma \in S^\omega$ is accepted by the Muller automaton if $\mathit{inf}_A(\sigma) \in \mathcal{Z}$.

When A_1, A_2 are two S-automata, then their product $A_1 \times A_2$ is defined as usual. When in the sequel we speak of a class $\mathcal{A}$ of automata, it is always taken for granted that $\mathcal{A}$ is defined by conditions on the set of states Z and the transition function r only, i.e. it is closed under redefinitions of the set of accepting states.

From any finitary language $L \subseteq S^*$ an infinitary language $\overrightarrow{L}$ is obtained by

$$\overrightarrow{L} := \{\sigma \in S^\omega \mid \sigma_i \in L \text{ i.o.}\}.$$

When $\mathcal{A}$ is a class of finite automata that is closed under products, then the following are equivalent for $\Sigma \subseteq S^\omega$ (cf.[21, Lemma 4.3]):

(a) Σ is accepted by some $A \in \mathcal{A}$ with a Muller acceptance condition.

(b) Σ is a Boolean combination of sets of the form $\overrightarrow{L(A)}$ with $A \in \mathcal{A}$.

(c) $\Sigma = \cup_{i=1}^k (\overrightarrow{L(A_i)} \cap S^\omega \setminus \overrightarrow{L(B_i)})$ for some $k \in \omega$, and $A_i, B_i \in \mathcal{A}$.

Finally, we associate with a nondeterministic transition system G with state set S the finite automaton $A_G = (S, S \cup \{\mathit{start}\}, \mathit{start}, r, S)$, where start is a state not in S, and $r(s, s') = s'$ if either $s = \mathit{start}$ and $s' \in S_0$, or $s \in S$ and $s' \in t(s)$. Then A_G accepts $\mathit{Run}_{\mathit{fin}}(G)$. Replacing S with $\mathcal{Z} := 2^S \setminus \emptyset$ yields a Muller automaton that accepts $\mathit{Run}(G)$.

4.2. Main Result

Suppose we are given a class of properties $\mathcal{C}$, and a fairness condition T, and now want to determine as large as possible a class $\mathcal{M} \subseteq \mathcal{M}_{FR(T,G)}$, such that (10) holds for all $\mu \in \mathcal{M}$. As the following example shows, it is in general not possible to obtain (10) for all $\mu \in \mathcal{M}_{FR(T,G)}$.

Example 4.1. Let S, G be as in Example 3.3. Let T be the condition of regular fairness (Example 2.7), and $\mathcal{C}$ the class of ω-regular properties (i.e. both T and $\mathcal{C}$ are defined by the class of all finite automata). $\mathit{FairRun}(T, G)$ contains both sequences starting with H and sequences starting with T. Let $\sigma \in \mathit{FairRun}(T, G)$ with $\sigma[1] = \mathsf{H}$. Let ϵ_σ be the unit probability point mass on σ. Then $\epsilon_\sigma \in \mathcal{M}_{FR(T,G)}$, and for $C = \mathsf{H}S^\omega \in \mathcal{C}$: $\epsilon_\sigma(C) = 1$, but $\mathit{FairRun}(T, G) \not\subseteq C$.

The probabilistic system ϵ_σ in the preceding example did not satisfy (10) because it possessed some very specific properties not shared by other (T, G)-fair implementations of G. These specific properties here derived from the fact that ϵ_σ is a measure that is highly concentrated on a small subset (indeed a singleton) of runs. This motivates the following definition, which allows us to exclude such highly concentrated measures.

Definition 4.2. Let G be a transition system, T a set of stopping strategies, μ a probability distribution on S^ω. We say that μ *has support* $\mathit{FairRun}(T, G)$, if $\mu \in \mathcal{M}_{FR(T,G)}$, and $\mu([\boldsymbol{s}]) > 0$ for all $\boldsymbol{s} \in \mathit{Run}_{\mathit{fin}}(G)$.

We can now formulate our completeness result.

Theorem 4.3. *Let $\mathcal{A}$ be a class of S-automata that is closed under products, and contains the automaton A_G for every transition system G with state set S. Let $T = \{\tau_A \mid A \in \mathcal{A}\}$. Let $\mathcal{C}$ be the class of all Boolean combinations of sets of the form $\overrightarrow{L(A)}$ with $A \in \mathcal{A}$. Then (10) holds for all transition systems G with state set S, all $C \in \mathcal{C}$, and all μ that have support* $\mathrm{FairRun}(T, G)$.

Before turning to the proof, we point out two interesting special cases of the theorem: when $\mathcal{A}$ is the class of counter free S-automata, then the class $\mathcal{C}$ defined in the theorem is just the class of LTL-definable properties (with S as the set of propositional variables; cf.[22, Theorem 6.7]), and T is the condition of α-fairness. Thus we regain the completeness result of Pnueli and Zuck **(year?)**. When $\mathcal{A}$ is the class of all finite S-automata, then $\mathcal{C}$ is the class of ω-regular properties, and T is the condition of regular fairness (Example 2.7).

Proof of Theorem 4.3: The right to left direction of (10) trivially holds by the condition that μ has support $\mathit{FairRun}(T, G)$. For the left to right direction, let $A = (S, Z, z_0, r, \mathcal{Z})$ be the Muller automaton that accepts C. We show that for $\sigma \in \mathit{FairRun}(T, G)$ then $U := \mathit{inf}_A(\sigma)$ has positive probability, i.e.

$$\mu(\{\sigma' \mid \mathit{inf}_A(\sigma') = U\}) > 0. \tag{11}$$

From $\mu(C) = 1$ it then follows that U must be an accepting set of states, and hence $\sigma \in C$.

To show (11), let $GA := A_G \times A$ be the product of A_G and A. We show that for a set $V \subseteq (S \cup \{\mathit{start}\}) \times Z$ of states in GA the following are equivalent:

(i) $V = \mathit{inf}_{GA}(\sigma)$ for some $\sigma \in \mathit{FairRun}(T, G)$.

(ii) V is a terminal strongly connected component (tscc) in GA that is reachable from the initial state by some $\boldsymbol{s} \in \mathit{Run}_{\mathit{fin}}(G)$.

(iii) $\mu(\{\sigma' \mid \mathit{inf}_{GA}(\sigma') = V\}) > 0$.

(i)⇒(ii): Let $\sigma \in \mathit{FairRun}(T, G)$ with $\mathit{inf}_{GA}(\sigma) = V$. V is a strongly connected component of GA by definition, so it only needs to be shown that V is terminal. Assume otherwise. Then there exists a state (s, z) in V from which a state $(s', z') \notin V$ is reachable by a transition labeled with s'. In particular, then $s' \in t(s)$ in G. Turn GA into a deterministic finite automaton by defining (s, z) to be its only accepting state. GA belongs to $\mathcal{A}$, and hence $\tau_{GA} \in T$. Furthermore $\sigma[\tau^i_{GA}] = s$ for infinitely many i, but $\sigma[\tau^i_{GA}, \tau^i_{GA} + 1] = ss'$ for at most finitely many i, a contradiction.

(ii)⇒(iii): Let $\boldsymbol{s}$ be as given in (ii). Then

$$[\boldsymbol{s}] \cap \mathit{Run}(G) \;=\; \{\sigma \mid \emptyset \neq \mathit{inf}_{GA}(\sigma) \subseteq V\} \;\supseteq\; [\boldsymbol{s}] \cap \mathit{FairRun}(T, G).$$

From μ having support $\mathit{FairRun}(T, G)$ it follows that $\mu(\{\sigma \mid \emptyset \neq \mathit{inf}_{GA}(\sigma) \subseteq V\} \cap \mathit{FairRun}(T, G)) > 0$. By the same argument as above, $\{\sigma \mid \mathit{inf}_{GA}(\sigma) \subsetneq V\} \cap \mathit{FairRun}(T, G) = \emptyset$, and (iii) follows.

(iii)⇒(i): From $\mu(\mathit{FairRun}(T, G)) = 1$ it follows that $\mu(\{\sigma \mid \mathit{inf}_{GA}(\sigma) = V\} \cap \mathit{FairRun}(T, G)) = \mu(\{\sigma \mid \mathit{inf}_{GA}(\sigma) = V\}) > 0$, and $\{\sigma \mid \mathit{inf}_{GA}(\sigma) = V\} \cap \mathit{FairRun}(T, G)$ is nonempty.

From the implication (i)⇒(iii) now (11) follows, because for $U = \mathit{inf}_A(\sigma)$ and $V = \mathit{inf}_{GA}(\sigma)$ we have $\{\sigma' \mid \mathit{inf}_A(\sigma') = U\} \supseteq \{\sigma' \mid \mathit{inf}_{GA}(\sigma') = V\}$, so that (11) follows from (iii). □

With the following corollary we extract from the proof of Theorem 4.3 the automata theoretic method for probabilistic verification [23, 24]. In the formulation of the corollary we use T_{GA} to denote the (finite) set of all stopping strategies τ_{GA} that are obtainable from the automaton GA as defined in the proof of Theorem 4.3 by various choices of accepting sets of states.

Corollary 4.4. *Let $C \subseteq S^\omega$ be recognized by a finite automaton A with states Z and Muller acceptance condition $\mathcal{Z}$. Let μ be a probability distribution on S^ω such that the following holds: there exists a nondeterministic transition system G with states S such that μ has support* $\mathrm{FairRun}(T_{GA}, G)$. *Then the following are equivalent*

(a) $\mu(C) = 1$

(b) *For all terminal strongly connected components V in GA that are reachable from the initial state by some $\boldsymbol{s} \in \mathrm{Run}_{\mathrm{fin}}(G)$:*

$$\{z \in Z \mid \exists s \in S : (s, z) \in V\} \in \mathcal{Z}.$$

Typically, the probabilistic system μ for which one wants to check the property C is given as a Markov chain. In this case the transition system G required by the corollary is simply given by the nonzero transition probabilities. Then condition (b) is effectively testable by constructing the automaton GA and checking its tscc's for membership in $\mathcal{Z}$.

Proof of Corollary 4.4: For fixed C and μ as in the corollary we obtain as in the proof of Theorem 4.3 the equivalence (ii)⇔(iii) (this is because only stopping strategies from T_{GA} are needed in the proof). We have $\mu(C) = 1$ iff for all

$U \subseteq Z$: $\mu(\{\sigma' \mid \inf_A(\sigma') = U\}) > 0$ implies $U \in \mathcal{Z}$. To obtain the equivalence (a)$\Leftrightarrow$(b) we now only have to note that $\mu(\{\sigma' \mid \inf_A(\sigma') = U\}) > 0$ iff there exists $V \subseteq (S \cup \{start\}) \times Z$ with $U = \{z \in Z \mid \exists s \in S : (s, z) \in V\}$ and $\mu(\{\sigma' \mid \inf_{GA}(\sigma') = V\}) > 0$. □

Theorem 4.3 shows that when both stopping strategies, and properties of sequences are expressed by finite automata, then there exists a natural balance between the resources used to define a particular concept of fairness T, and the richness of the class $\mathcal{C}$ for which completeness is obtained. What happens when one goes beyond finite automata? Especially, what completeness results do we obtain for computable fairness? This is an interesting and mostly open problem. We can pose it more pointedly by defining $\mathcal{C}_{cf}(S)$ to be the set of all $C \subseteq S^\omega$ such that (10) holds for all transition systems G and all μ with support $\mathit{FairRun}(\mathcal{T}(S), G)$. The question then is for alternative characterizations of $\mathcal{C}_{cf}(S)$. We know from Theorem 4.3 that all ω-regular $C \subseteq S^\omega$ belong to $\mathcal{C}_{cf}(S)$. It is easy to construct C that are not ω-regular but also belong to $\mathcal{C}_{cf}(S)$. As the following example shows, however, there exist limits of context-free languages which already do not belong to $\mathcal{C}_{cf}(S)$.

Example 4.5. Let S, G be as in Example 3.3. Define $L := \{\boldsymbol{s} \in \{\mathsf{H}, \mathsf{T}\}^* \mid |\{i \mid \boldsymbol{s}[i] = \mathsf{H}\}| = |\{i \mid \boldsymbol{s}[i] = \mathsf{T}\}|\}$. L is a context-free language. Let $C := \overrightarrow{L}$. Now consider $T = \mathcal{T}(S)$, and the class $\mathcal{M}$ of Markov chains on G.

If a completeness held for T, $\mathcal{M}$, and C, then this, in particular, would entail the robustness property $\mu(C) = 1 \Leftrightarrow \mu'(C) = 1$ for all $\mu, \mu' \in \mathcal{M}$. This, however, is not the case, as we have $\mu(C) = 1$ for μ defined by transition probabilities $1/2$ for all transitions, whereas $\mu'(C) = 0$ for μ' defined e.g. by transition probabilities $p(\mathsf{H}, \mathsf{T}) = p(\mathsf{T}, \mathsf{T}) = 1/3$, $p(\mathsf{H}, \mathsf{H}) = p(\mathsf{T}, \mathsf{H}) = 2/3$. It thus follows that the condition of computable fairness is not sufficient to obtain a completeness result for limits of context-free properties and the class of Markov chains.

5. Extending the System Model

5.1. Mixed Models

Many previous studies considered questions of fairness and its relation to probabilistic behavior on the basis of system models that combine probabilistic and nondeterministic behavior [3, 1, 24, 25]. As observed by Vardi **(year?)**, these models are mostly variants of the classical Markov Decision Process model. They are appropriate e.g. for modeling the interaction of a probabilistic system with a nondeterministic environment, or decisions of an agent in a probabilistic environment.

One of the simplest and clearest model of this type are Vardi's **(year?)** *reactive Markov chains*. A reactive Markov chain is a hybrid of a nondeterministic transition system and a probabilistic system in our sense: it is given by a set of states S, which is partitioned into a set of nondeterministic states N, and a set of probabilistic states P. With each state $s \in N$ is associated a set $t(s) \subseteq S$ of possible successor states, and with each state $s \in P$ are associated transition probabilities $p(s, s')$ ($s' \in S$, cf. Example 3.5).

A reactive Markov chain can be interpreted as a model for the interaction of a non-deterministic and a probabilistic player. There are (at least) two different perspectives under which one can consider fairness properties of such a system, depending on whether one wants to investigate fairness of the behavior of the non-deterministic or the probabilistic player.

If we focus on the non-deterministic player, then the question will be what kind of global system behavior can be guaranteed by suitable fairness assumptions on the non-deterministic player, and how these relate to possible probabilistic implementations also for the non-deterministic player. From the point of view we have adopted in this paper, a reactive Markov chain then can be seen as a partial implementation of a fully non-deterministic system. Our definitions and results can be relativized to such partial implementations: stopping strategies need to be restricted to only stop at non-deterministic states (since they are needed for specifying fairness conditions for the non-deterministic player). Probabilistic systems μ must be constrained to be consistent with the given partial implementation, i.e. must satisfy $\mu(\boldsymbol{s}ss' \mid \boldsymbol{s}s) = p(s, s')$ for all $\boldsymbol{s} \in S^*$, $s \in P$, $s' \in S$. Our main results Theorems 3.4 and 4.3 can be generalized to allow for such restrictions on admissible stopping strategies and probabilistic systems (for the generalization of Theorem 4.3 one will also have to limit the run properties C to those properties that only depend on the embedded sequence of nondeterministic states).

Alternatively, one can also focus on the probabilistic player, treat the non-deterministic player as completely unknown, and ask, e.g. what kind of system properties can be inferred from the fairness properties of the probabilistic player (this is the perspective adopted e.g. in [3]). Each strategy of the non-deterministic player induces a fully probabilistic system (with 0/1-valued transition probabilities − possibly history dependent − from non-deterministic states). Our results can be applied in this setting by restricting admissible stopping strategies to only stop at probabilistic states, and by considering all possible probabilistic systems μ obtained from possible strategies of the non-deterministic player.

Many other types of systems proposed in the literature can be reduced to the reactive Markov chain model. Pnueli & Zuck **(year?)**, for example, use a system model in which the transition from one system state to the next is composed of three separate moves: first there is a non-deterministic choice of one of several available transitions (which, here, do not yet determine the successor state, but can be thought of as a 'transition type' or 'transition label'), then there is a probabilistic choice of a *mode* for the given transition, and finally another non-deterministic choice of an actual successor state from a set of possible successors, which is determined by the chosen transition and mode. This model can be represented as a reactive Markov chain by introducing explicit state representations for the three component moves of a state transition.

5.2. Infinite State Systems

While our basic definitions of stopping strategies and fairness could easily be extended to infinite state spaces, it is rather unclear that for infinite systems they would still reflect reasonable and relevant conditions on system behavior (in an infinite state space one may not expect to see any states recurring infinitely often, so that fairness conditions in the form of (1) would become vacuous). However, Example 3.7 illustrates how our concepts and results can still be relevant for an infinite state system: for the purpose of a particular analysis one is often interested in a "finite state abstraction" of an infinite state space, which is obtained by grouping the infinitely many states into finitely many equivalence classes. For probabilistic systems such an abstraction will usually destroy Markov properties, i.e. an (infinite state) stationary Markov chain will induce a probabilistic system on the reduced state space that is neither Markov nor stationary (as in Example 3.7). It is therefore important to note that our results also include non-Markovian probabilistic systems: to some extent this makes them applicable to infinite state systems.

6. Related Work and Conclusion

Connections between our results and those of Baier and Kwiatkowska [1] have already been pointed out in Section 2.2. Varacca and Völzer [15] define fairness properties as *topologically large* sets of runs, and establish for ω-regular properties a correspondence with *probabilistically large* sets (i.e. sets of probability 1). As in the work of Baier and Kwiatkowska, the results are obtained only for ϵ-bounded probabilistic systems. De Alfaro [4], too, mostly considers probabilistic systems whose fairness properties are determined by a global ϵ-bound on transition probabilities. Such systems are called *probabilistically fair* by de Alfaro, and it is shown that probabilistically fair systems possess some strong and robust fairness properties (notably invariance of fairness under synchronous composition). These results can be seen as a special instance of the right to left direction of our Theorem 3.4.

Jurdzinski, Kupferman and Henzinger [26] have investigated connections between probabilistic and nondeterministic systems from a somewhat different perspective. Their main goal is to reduce certain decision problems for probabilistic games to decision problems for nondeterministic games. The basic motivation, thus, is similar to the motivation for our study of completeness in Section 4. However, the main issue addressed by Jurdzinski et al. [26] is not simply an elimination of probabilistic components from the system model, but the elimination of synchronous transitions: the probabilistic games considered contain simultaneous moves by the two players, and the main problem solved by Jurdzinski et al. is the reduction to a game structure where two players take alternating turns.

None of these previous works have investigated the connections between fairness and the classical definitions of randomness, which is our first main contribution. Furthermore, we have introduced the notion of a (T, G)-divergent probabilistic system, which enables us to extend the analysis of fairness properties of probabilistic systems from previously studied ϵ-bounded systems. Example 3.7 illustrates how this generalization can be useful in the analysis of infinite state systems, where global ϵ-bounds can often be an unrealistic assumption.

Our investigation was based on the simplest possible system models on the basis of which fundamental questions concerning nondeterministic, probabilistic, and fair behavior can be studied. Benefiting from the simplicity of these models, we obtained very succinct proofs for our results. While some additional work is required to lift these basic results to more complex systems models and apply them to more specific application problems in system analysis, we believe that they capture key insights and key arguments, which can be adapted to a wide variety of more specialized contexts.

Acknowledgments

The author thanks Lenore Zuck for her careful reading of an earlier version of this paper. Her comments, as well as comments provided by an anonymous reviewer, helped a great deal to improve the clarity and presentation of this paper.

References

[1] C. Baier, M. Kwiatkowska, On the verification of qualitative properties of probabilistic processes under fairness constraints, Information Processing Letters 66 (1998) 71–79.

[2] A. Pnueli, On the extremely fair treatment of probabilistic algorithms, in: Proceedings of the 18th IEEE Symposium on Foundation of Computer Science, 1983.

[3] A. Pnueli, L. D. Zuck, Probabilistic verification, Information and Computation 103 (1993) 1–29.

[4] L. de Alfaro, From fairness to chance, Electronic Notes in Theoretical Computer Science 22.

[5] P. Martin-Löf, The definition of random sequences, Information and Control 9 (1966) 602–619.

[6] C. P. Schnorr, A unified approach to the definition of random sequences, Math. Systems Theory 5 (1971) 246–258.

[7] H. Gaifman, M. Snir, Probabilities over rich languages, testing and randomness, Journal of Symbolic Logic 47 (3) (1982) 495–548.

[8] L. Lamport, "Sometime" is sometimes "Not Never", in: Conference Record of the Seventh ACM Symposium on Principles of Programming Languages (POPL '80), ACM Press, 1980.

[9] D. Lehmann, A. Pnueli, J. Stavi, Impartiality, justice and fairness: the ethics of concurrent termination, in: S. Even, O. Kariv (Eds.), Proceedings of ICALP-81, Vol. 115 of Lecture Notes in Computer Science, Springer, 1981, pp. 264–277.

[10] Z. Manna, A. Pnueli, Completing the temporal picture, in: G. Ausiello, M. Dezani-Ciancaglini, S. R. D. Rocca (Eds.), Automata, languages, and programming : 16th international colloquim (ICALP 89, 1989, pp. 534–558.

[11] K. R. Apt, N. Francez, S. Katz, Appraising fairness in languages for distributed programming, Distributed Computing 2 (1988) 226–241.

[12] B. Aminof, T. Ball, O. Kupferman, Reasoning about systems with transition fairness, in: F. Baader, A. Voronkov (Eds.), Proceedings of LPAR-04, Vol. 3452 of LNCS, 2005, pp. 194–208.

[13] L. Lamport, Fairness and hyperfairness, Distributed Computing 13 (2000) 239–245.

[14] H. Völzer, D. Varacca, E. Kindler, Defining fairness, in: M. Abadi, L. de Alfaro (Eds.), Proceedings of CONCUR-05, Vol. 3653 of LNCS, 2005, pp. 458–472.

[15] D. Varacca, H. Völzer, Temporal logics and model checking for fairly correct systems, in: LICS '06: Proceedings of the 21st Annual IEEE Symposium on Logic in Computer Science, IEEE Computer Society, Washington, DC, USA, 2006, pp. 389–398. doi:http://dx.doi.org/10.1109/LICS.2006.49.

[16] M. Jaeger, Fairness, computable fairness and randomness, in: Proceedings of the 2nd International Workshop on Probabilistic Methods in Verification (PROBMIV-99), Technical Report CSR-99-8, School of Computer Science, University of Birmingham, Eindhoven, Netherlands, 1999, pp. 57–66.

[17] R. von Mises, Grundlagen der wahrscheinlichkeitsrechnung, Mathematische Zeitschrift 5 (1919) 52–99.

[18] M. van Lambalgen, The axiomatization of randomness, The Journal of Symbolic Logic 55 (3) (1990) 1143–1167.

[19] M. Li, P. Vitányi, An Introduction to Kolmogorov Complexity and Its Applications, Springer, 1997.

[20] P. Billingsley, Probability and Measure, Wiley, 1986.

[21] W. Thomas, Automata on infinite objects, in: J. van Leeuwen (Ed.), Handbook of Theoretical Computer Science, Vol. 2, Elsevier/MIT Press, 1990.

[22] E. Emerson, Temporal and modal logic, in: J. van Leeuwen (Ed.), Handbook of Theoretical Computer Science, Vol. 2, Elsevier, 1990.

[23] C. Courcoubetis, M. Yannakakis, The complexity of probabilistic verification, Journal of the ACM 42 (4) (1995) 857–907.

[24] M. Y. Vardi, Probabilistic linear-time model checking: an overview of the automata-theoretic approach, in: J.-P. Katoen (Ed.), Formal methods for real-time and probabilstic systems (ARTS-99) : 5th International AMAST Workshop, Vol. 1601 of LNCS, 1999.

[25] C. Baier, M. G. er, Recognizing ω-regular languages with probabilistic automata, in: LICS '05: Proceedings of the 20th Annual IEEE Symposium on Logic in Computer Science, 2005, pp. 137–146.

[26] M. Jurdzinski, O. Kupferman, T. Henzinger, Trading probability for fairness, in: Proceedings of CSL-02, 2002.