Advanced Algorithm Design and Analysis (Lecture 11)

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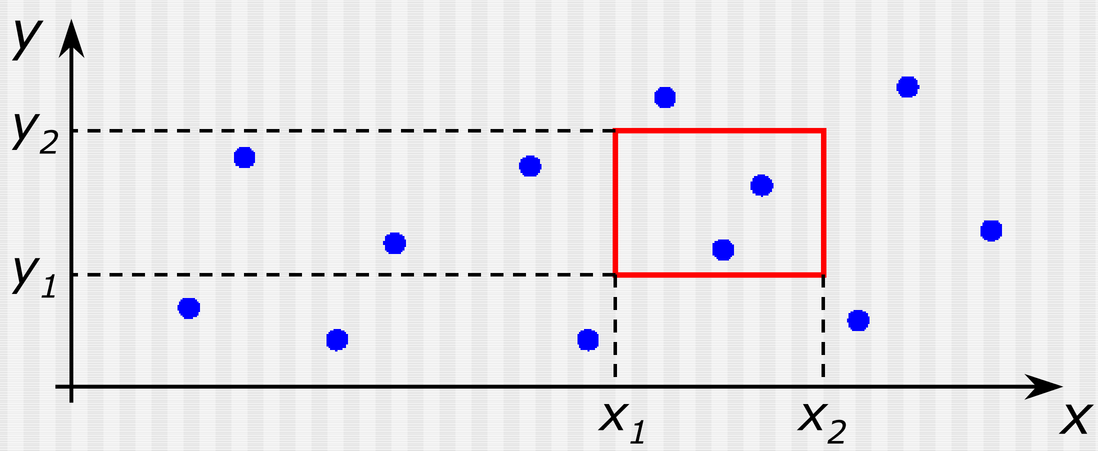
Range Searching in 2D

- Main goals of the lecture:
  - *to understand and to be able to analyze*
    - *the **kd-trees** and the **range trees**;*
  - *to see how data structures can be used to trade the space used for the running time of queries*

Range queries

- *How do you efficiently find points that are inside of a rectangle?*
  - **Orthogonal range** query ($[x_1, x_2], [y_1, y_2]$): find all points (x, y) such that $x_1<x<x_2$ and $y_1<y<y_2$
  - Useful also as a multi-attribute database query

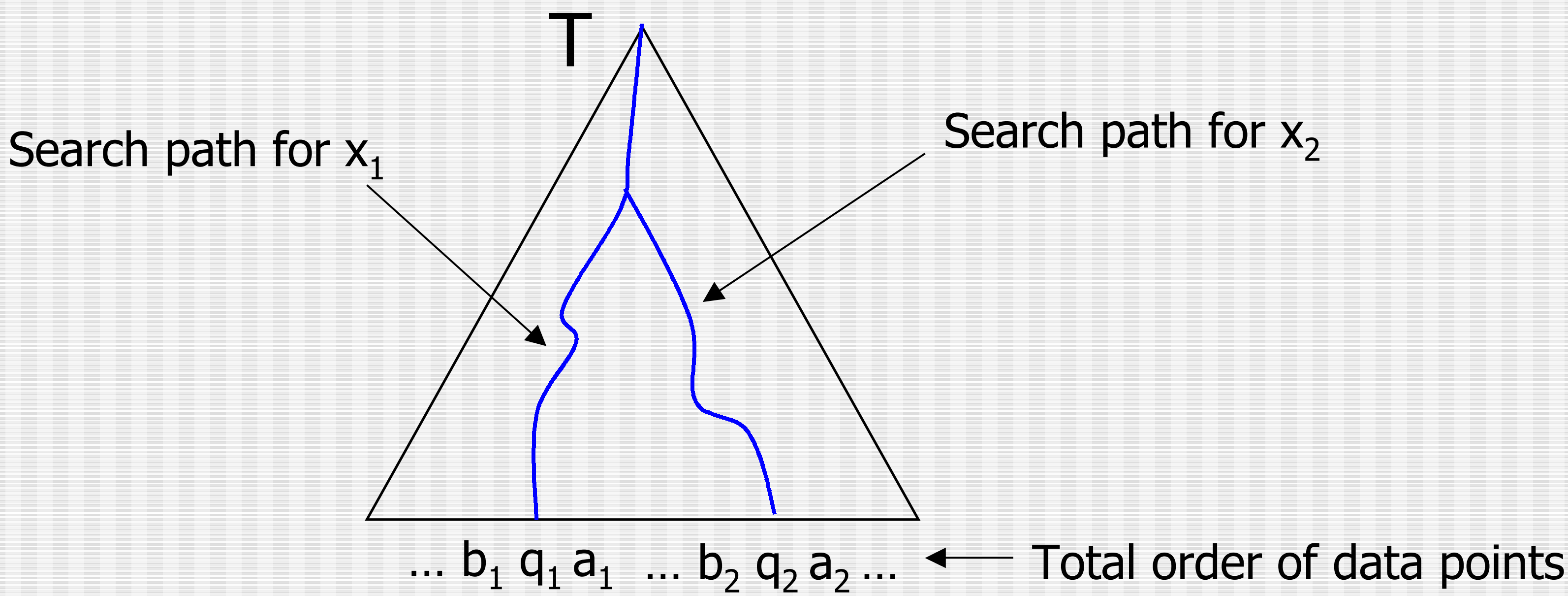


Preprocessing

- *How much time such a query would take?*
- *Rules of the game:*
  - We **preprocess** the data into a data structure
  - Then, we perform queries and updates on the data structure
  - Analysis:
    - Preprocessing time
    - Efficiency of queries (and updates)
    - The size of the structure
  - *Assumption*: no two points have the same *x*-coordinate (the same is true for *y*-coordinate).

1D range query

- *How do we do a 1D range query* $[x_1, x_2]$?
  - Balanced BST where all data points are stored in the leaves
    - *The size of it?*
  - *Where do we find the answer to a query*?



1D range query

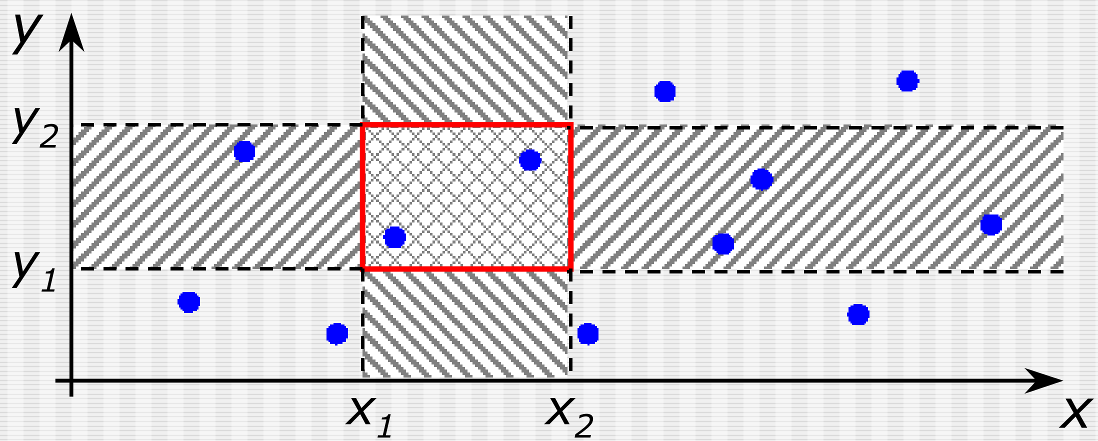
- *How do we find all these leaf nodes*?
  - A possibility: have a linked list of leaves and traverse from q_1 to q_2
    - but, will not work for more dimensions…
  - Sketch of the algorithm:
    - Find **the split node**
    - Continue searching for x1, report all right-subtrees
    - Continue searching for x2, report all left-subtrees
    - When leaves q_1 and q_2 are reached, check if they belong to the range
  - *Why is this correct*?

Analysis of 1D range query

- *What is the worst-case running time of a query*?
  - It is *output-sensitive*: two traversals down the tree plus the $O(k)$, where k is the number of reported data points: $O(\log n + k)$
- *What is the time of construction*?
  - Sort, construct by dividing into two, creating the root and conquering the two parts recursively
  - $O(n \log n)$
- Size: $O(n)$

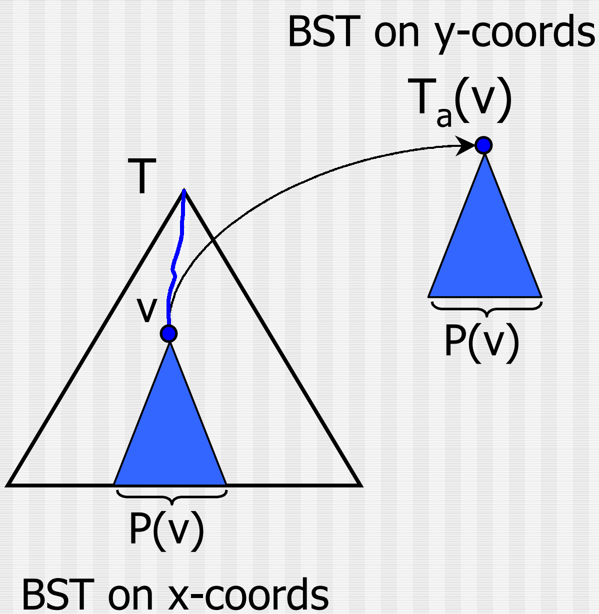
2D range query

- *How can we solve a 2D range query*?
  - *Observation* – 2D range query is a conjunction of two 1D range queries: $x_1<x<x_2$ **and** $y_1<y<y_2$
  - Naïve idea:
    - have two BSTs (on *x*-coordinate and on *y*-coordinate)
    - Ask two 1D range queries
    - Return the intersection of their results
  - *What is the worst-case running time (and when does it happen)? Is it output-sensitive?*



Range tree

- *Idea: when performing search on x-coordinate, we need to start filtering points on y-coordinate earlier!*
  - **Canonical subset** $P(v)$ of a node v in a BST is a set of points (leaves) stored in a subtree rooted at v
  - **Range tree** is a multi-level data structure:
    - The main tree is a BST T on the x-coordinate of points
    - Any node v of T stores a pointer to a BST $T_a(v)$ (**associated structure** of v), which stores canonical subset $P(v)$ organized on the y-coordinate
    - 2D points are stored in all leaves!



Querying the range tree

- *How do we query such a tree*?
  - Use the *1DRangeSearch* on *T,* but replace *ReportSubtree*(w) with *1DRangeSearch*(T_a(w), y_1, y_2)
- *What is the worst-case running time?*
  - Worst-case: We query the associated structures on all nodes on the path down the tree
  - On level *j,* the depth of the associated structure is

$$\log \frac{n}{2^j} = \log n - j$$

  - Total running time: $O(\log^2 n + k)$

Size of the range tree

- *What is the size of the range tree*?
  - At each level of the main tree associated structures store all the data points once (with constant overhead) (*Why*?) : $O(n)$
  - There are $O(\log n)$ levels
  - Thus, the total size is $O(n \log n)$

Building the range tree

- *How do we efficiently build the range tree*?
  - Sort the points on x and on y (two arrays: X,Y)
  - Take the median v of X and create a root, build its associated structure using Y
  - Split X into sorted X_L and X_R, split Y into sorted Y_L and Y_R (s.t. for any $p \in X_L$ or $p \in Y_L$, $p.x < v.x$ and for any $p \in X_R$ or $p \in Y_R$, $p.x \geq v.x$)
  - Build recursively the left child from X_L and Y_L and the right child from X_R and Y_R
- *What is the running time of this*?
  - $O(n \log n)$

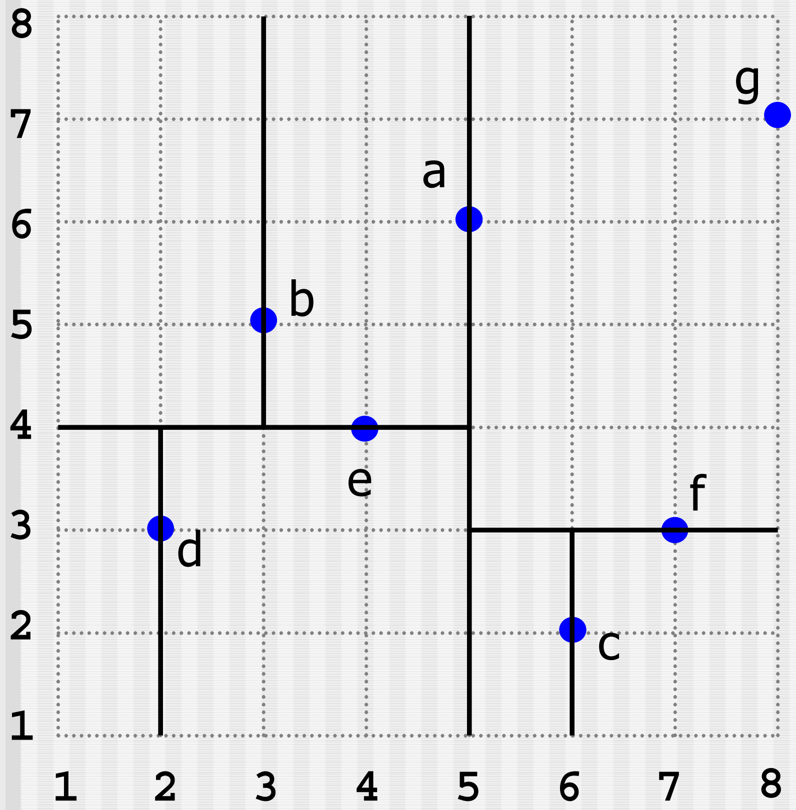
Range trees: summary

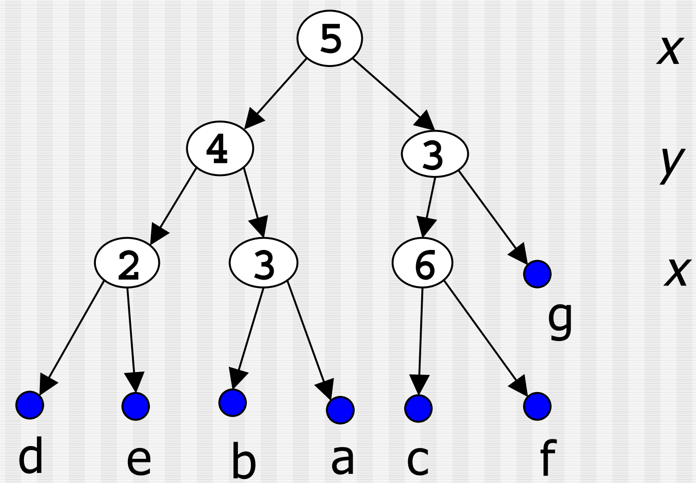
- **Range trees**
  - Building (preprocessing time): $O(n \log n)$
  - Size: $O(n \log n)$
  - Range queries: $O(\log^2 n + k)$
- Running time can be improved to $O(\log n + k)$ without sacrificing the preprocessing time or size
  - Layered range trees (uses *fractional cascading*)
  - Priority range trees (uses *priority search trees* as associated structures)

Kd-trees

- *What if we want linear space*?
  - Idea: **partition trees** – generalization of binary search trees
  - Kd-tree: a binary tree
    - Data points are at leaves
    - For each internal node *v*:
      - *x*-coords of left subtree ≤ *v* < *x*-coords of right subtree, **if** depth of v is *even* (*split with vertical line*)
      - *y*-coords of left subtree ≤ v < *y*-coords of right subtree, **if** depth of v is *odd* (*split with horizontal line*)
  - Space: $O(n)$ – points are stored once.

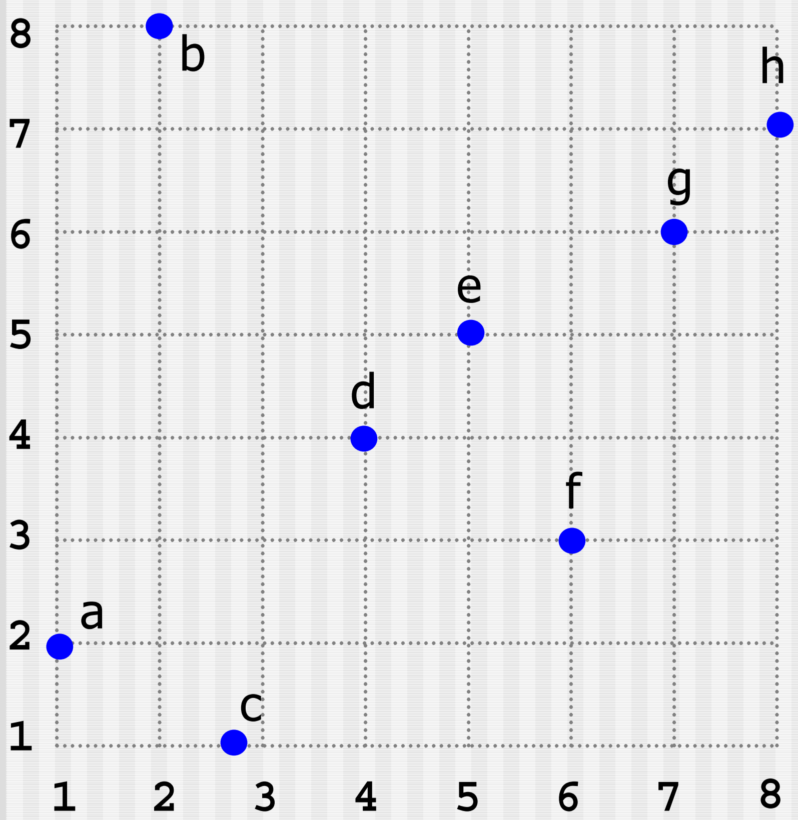
Example kd-tree





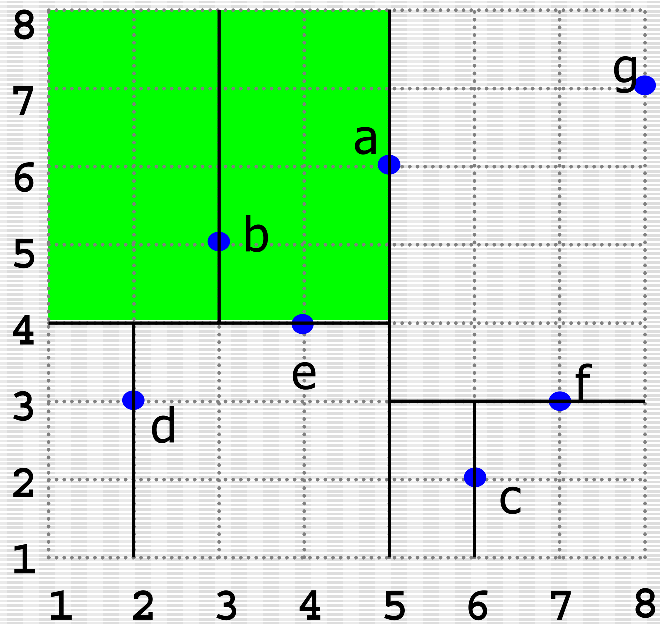
Draw a kd-tree

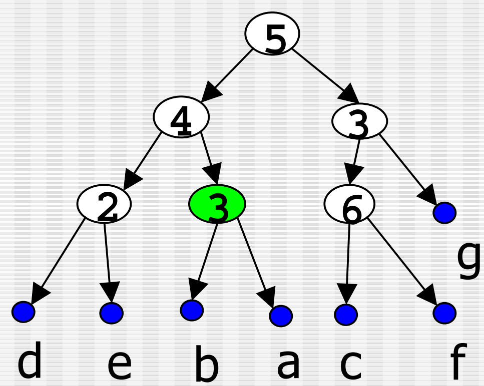
- *Draw a kd-tree storing the following data points*



Querying the kd-tree

- *How do we answer a range query*?
  - *Observation*: Each internal node *v* corresponds to a *region*(*v*) (where all its children are included).
  - We can maintain *region*(*v*) as we traverse down the tree





Querying the kd-tree

- *The range query algorithm (query range R)*:
  - If *region*(*v*) **does not intersect** *R,* do not go deeper into the subtree rooted at *v*
  - If *region*(*v*) is **fully contained** in *R,* report all points in the subtree rooted at v
  - If *region*(*v*) **only intersects** with *R*, go recursively into *v*'s children.

Analysis of the search alg.

- *What is the worst-case running time of the search*?
  - Traversal of subtrees *v*, such that *region*(*v*) is fully contained in *R* adds up to *O*(*k*).
  - We need to find the number of regions that intersect *R* – the regions which are crossed by some border of *R*
    - As an upper bound for that, let's find how many regions a crossed by a vertical (or horizontal) line
    - *What recurrence can we write for it*?

$$T(n) = 2 + 2T(n/4)$$

- Solution: $O(\sqrt{n})$ Total time: $O(\sqrt{n} + k)$

Building the kd-tree

- *How do we build the kd-tree*?
  - Sort the points on x and on y (two arrays: X,Y)
  - Take the median v of X (if depth is even) or Y (if depth is odd) and create a root
  - Split X into sorted X_L and X_R, split Y into sorted Y_L and Y_R, s.t.
    - for any $p \in X_L$ or $p \in Y_L$, $p.x < v.x$ (if depth is even) or $p.y < v.y$ (if depth is odd)
    - for any $p \in X_R$ or $p \in Y_R$, $p.x \geq v.x$ (if depth is even) or $p.y \geq v.y$ (if depth is odd)
  - Build recursively the left child from X_L and Y_L and the right child from X_R and Y_R

- *What is the running time of this*?
  - $O(n \log n)$

Kd-trees: summary

- Kd-tree:
  - Building (preprocessing time): $O(n \log n)$
  - Size: $O(n)$
  - Range queries: $O(\sqrt{n} + k)$

Quadtrees

- *Quadtree – a four-way partition tree*
  - ***region*** quadtrees vs. ***point*** quadtrees
    - *kd-trees can also be point or region*
  - Linear space
  - Good average query performance