Advanced Algorithm Design and Analysis (Lecture 2)

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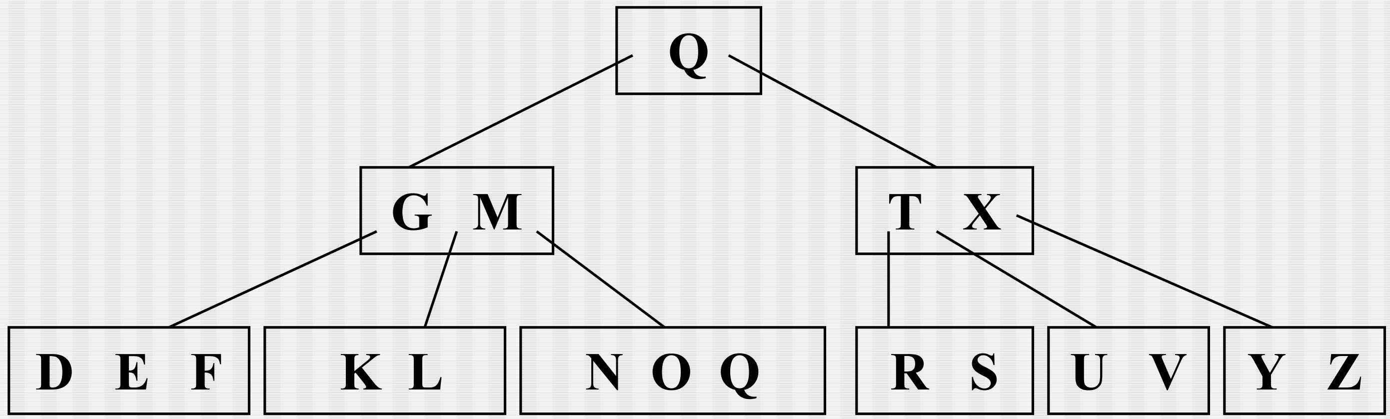
External Mem. DS & Algs

- B-trees: insertion and deletion
- External-memory sorting
- Goals of the lecture:
  - *to understand the algorithms of B-tree and its variants and to be able to analyze them;*
  - *to understand how the different versions of* ***merge-sort*** *derived algorithms work in external memory and to be able to compare their efficiency;*
  - *to understand why the amount of available main-memory is an important parameter for the efficiency of external-memory algorithms.*

B-trees: Insert

- Insertion is always performed at the leaf level
- Let's do an example (t = 2):
  - Insert: *H, J, P*

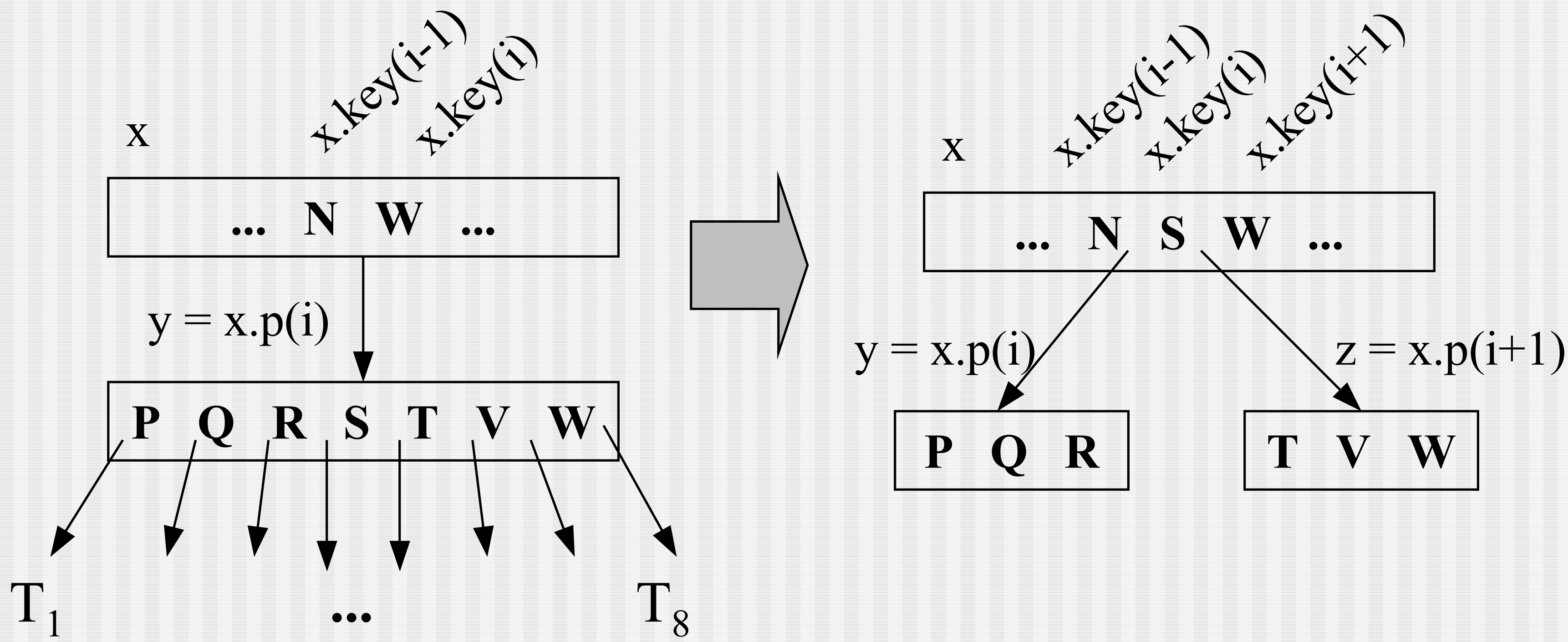


Splitting Nodes I

- Nodes fill up and reach their maximum capacity $2t - 1$
- Before we can insert a new key, we have to “make room,” i.e., split a node

Splitting Nodes II

- Result: one key of *x* moves up to parent + 2 nodes with *t*-1 keys
  - How many I/O operations?



Insert I

- Skeleton of the algorithm:
  - *Down-phase:* recursively traverse down and find the leaf
  - Insert the key
  - *Up-phase:* if necessary, split and propagate the splits up the tree
- Assumptions:
  - In the *down-phase* pointers to traversed nodes are saved in the stack. Function *parent*(*x*) returns a parent node of *x* (pops the stack)
  - *split*(*y*:Bnode):(*zk*:key_t, *z*:Bnode)

Insert II

```
DownPhase(x,k)
i ← 1
while i ≤ x.n() and k > x.key(i)
    i ← i+1
if x.leaf() then
    return x
    else DiskRead(x.p(i))
         return DownPhase(x.p(i),k)
```

```
Insert(T,k)
x ← DownPhase(T.root(), k)
UpPhase(x, k, nil)
```

Insert III

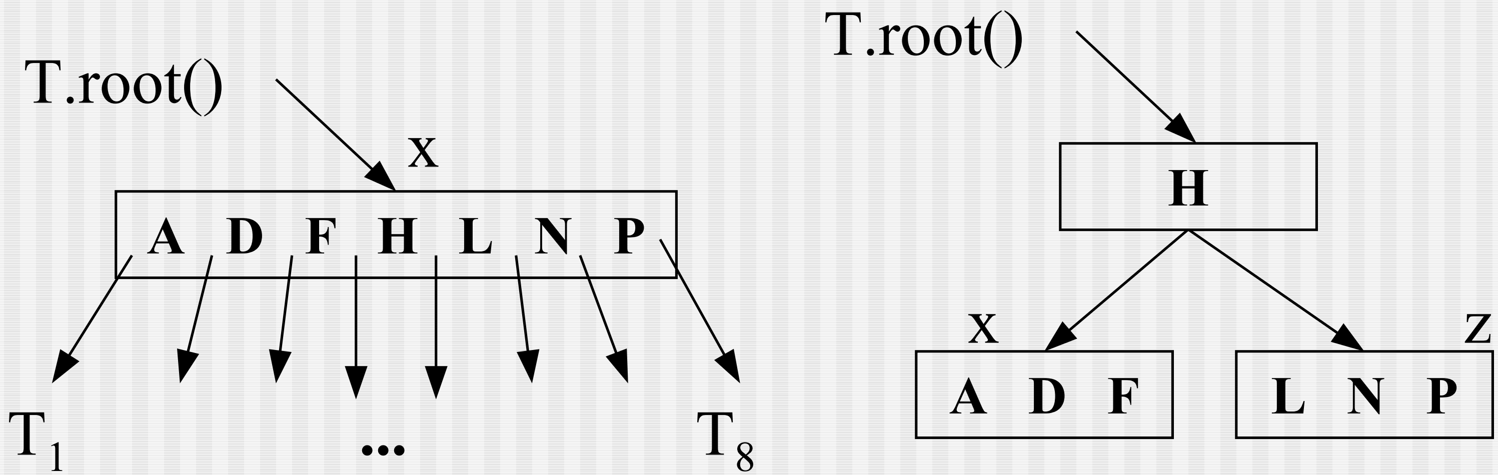
```
UpPhase(x,k,p)
if x.n() = 2t-1 then
   (zk,z) ← split(x)
   if k ≤ zk then InsertIntoNode(x,k,p)
             else InsertIntoNode(z,k,p)
   if parent(x)=nil then (Create new root)
   else UpPhase(parent(x),zk,z)
else InsertIntoNode(x,k,p)
```

InsertIntoNode(x,k,p)

Inserts the hey *k* and the following pointer *p* (if not *nil*) into the sorted order of keys of *x*, so that all the keys before *k* are smaller or equal to *k* and all the keys after *k* are greater than *k*

Splitting the Root

- Splitting the root requires the creation of a new root



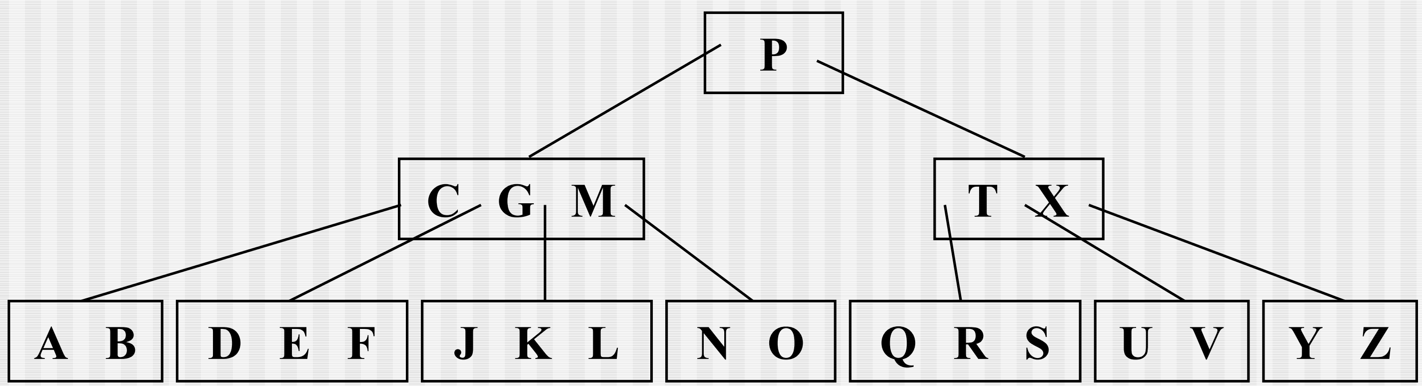
- The tree grows at the top instead of the bottom

One/Two Phase Algorithms

- Running time: $O(h) = O(log_B n)$
- Insert could be done in one traversal down the tree (by splitting all full nodes that we meet, "just in case")
- Disadvantage of the two-phase algorithm:
  - Buffer of $O(h)$ pages is required

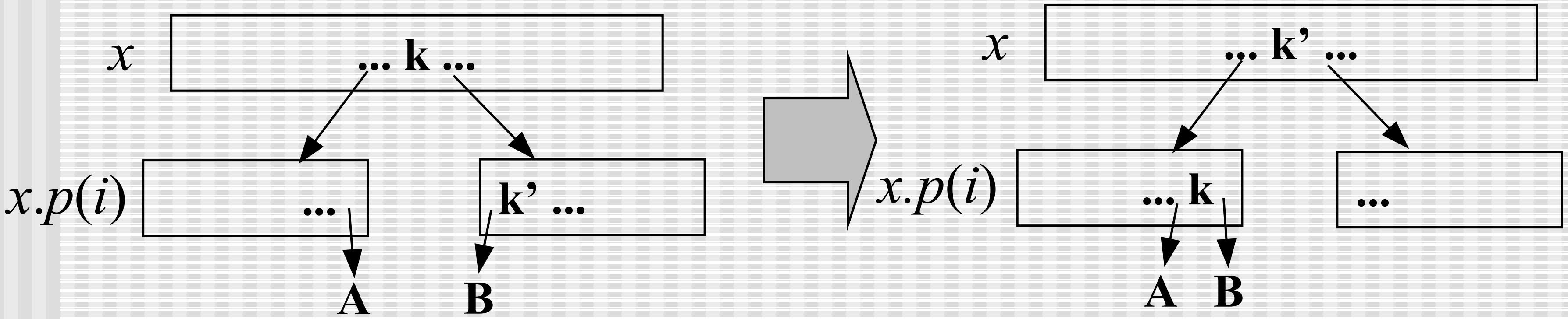
Deletion

- Case 1: A key *k* is in a non-leaf node
  - Delete its predecessor (which is always in a leaf, thus case 2) and put it in *k's* place.
- Case 2: A key is in a leaf-node:
  - Just delete it and handle under-full nodes
- Try: delete *M, B, K* (t =3)

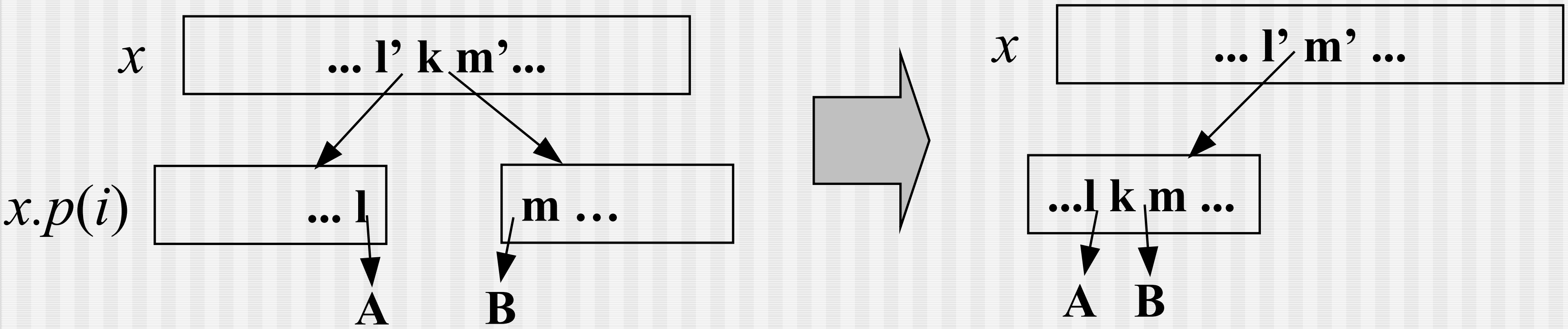


Handling Under-full Nodes

- Distributing:



- Merging:

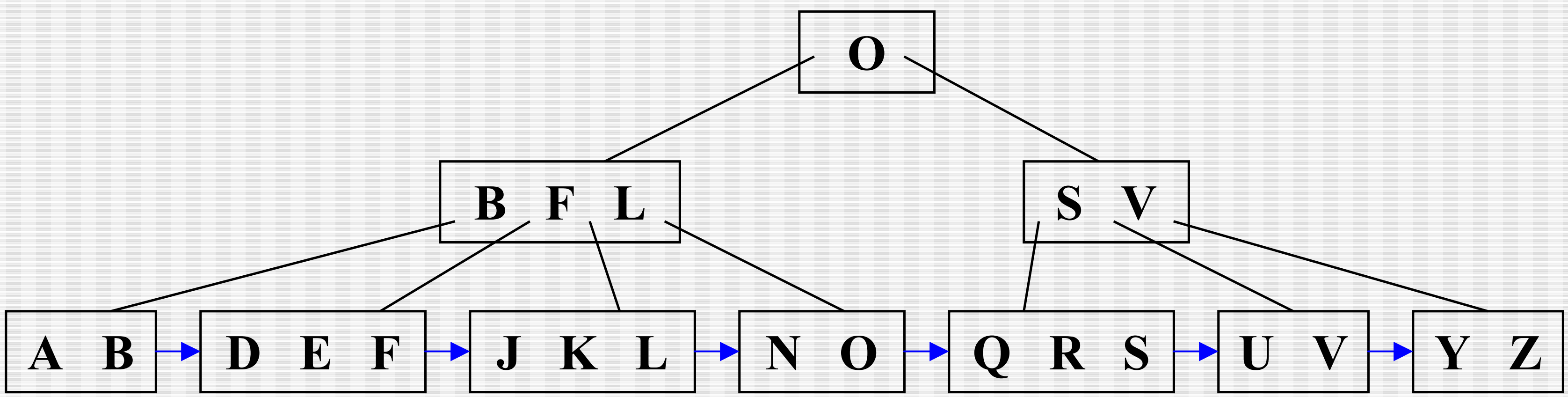


Sequential access

- Other useful ADT operator: *successor*
  - For example, range queries: *find all accounts with the amount in the range* [100K – 200K].
  - How do you do that in B-trees?

B+-trees I

- B+-trees is a variant of B-trees:
  - All data keys are in leaf nodes
    - The split does not *move* the middle key to the parent, but *copies* it to the parent!
  - Leaf-nodes are connected into a (doubly) linked list



B^+-trees II

- Lets draw a B^+-tree (t =2):
  - *A, B, C, D, E, F, G, H, I, J, K*
- How is the range query performed?
  - Compare with the B-tree

External-Memory Sorting

- External-memory algorithms
  - When data do not fit in main-memory
- External-memory sorting
  - Rough idea: sort peaces that fit in main-memory and “merge” them
- Main-memory merge sort:
  - The main part of the algorithm is Merge
  - Let’s merge:
    - 3, 6, 7, 11, 13
    - 1, 5, 8, 9, 10

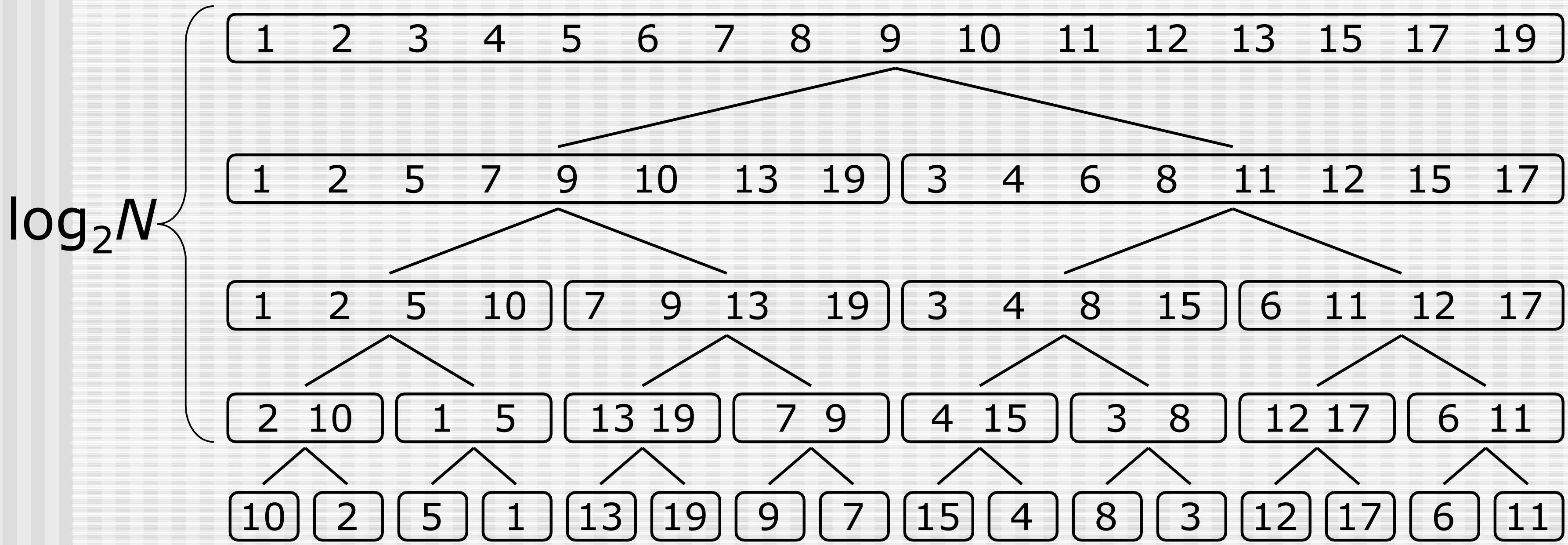
Main-Memory Merge Sort

```
Merge-Sort(A)
if length(A) > 1 then
    Copy the first half of A into array A1
    Copy the second half of A into array A2
    Merge-Sort(A1)
    Merge-Sort(A2)
    Merge(A, A1, A2)
```

Divide (lines: copy halves into A1 and A2)
Conquer (lines: Merge-Sort(A1), Merge-Sort(A2))
Combine (line: Merge(A, A1, A2))

- Running time?

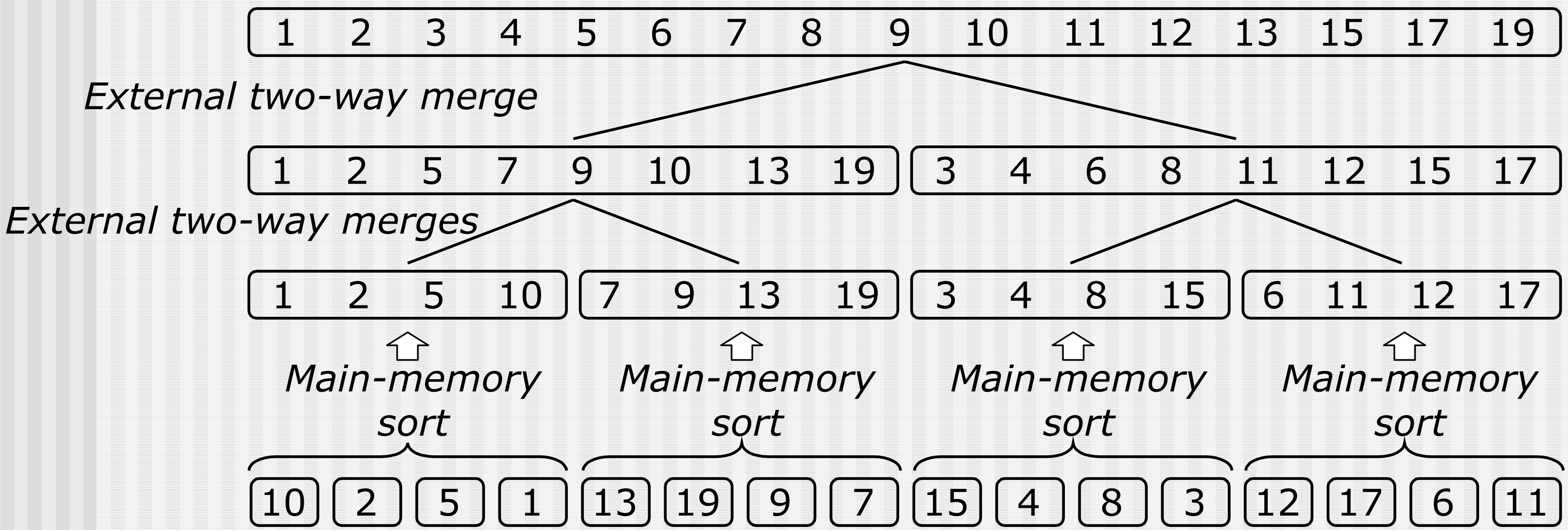
Merge-Sort Recursion Tree



- In each level: merge *runs* (sorted sequences) of size x into runs of size $2x$, decrease the number of runs twofold.
- What would it mean to run this on a file in external memory?

External-Memory Merge-Sort

- Idea: increase the size of initial runs!
  - Initial runs – the size of available main memory (*M* data elements)

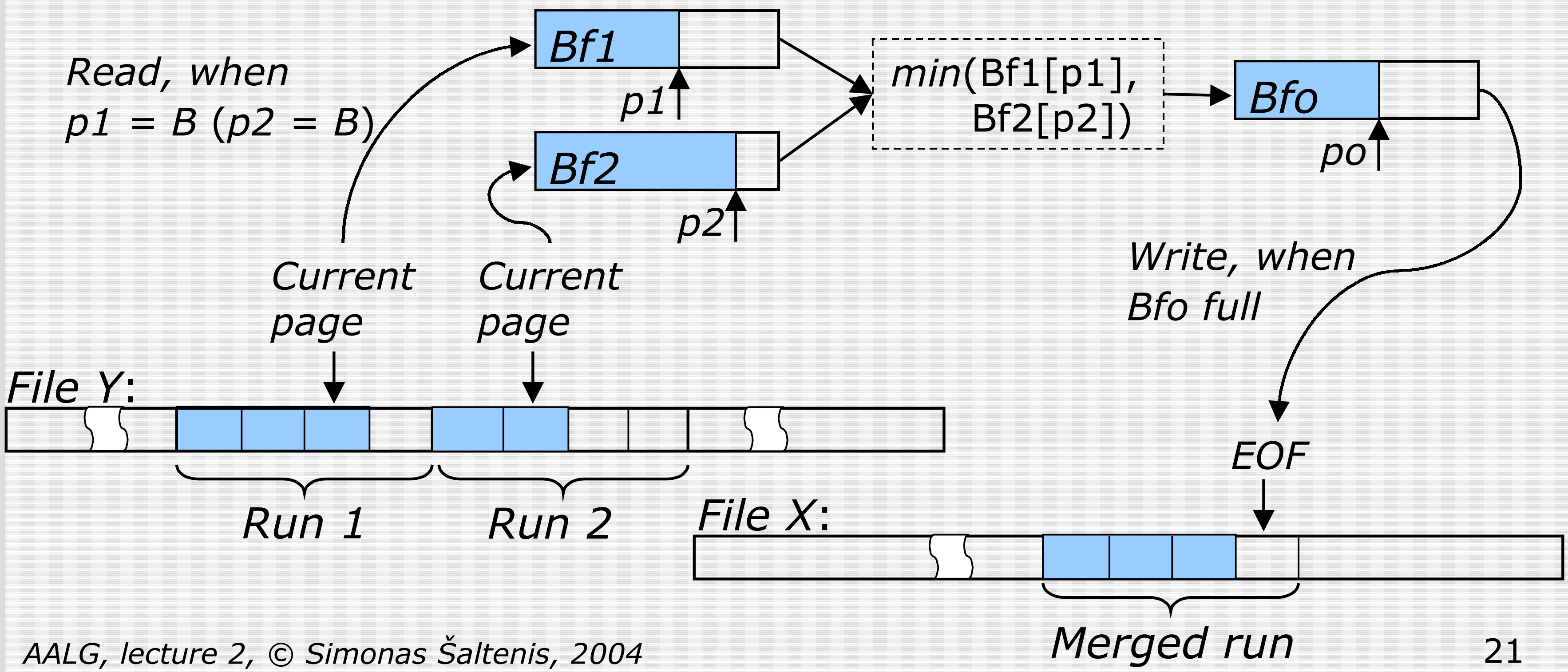


External-Memory Merge Sort

- Input file *X*, empty file Y
- *Phase* 1: Repeat until end of file *X*:
  - Read the next *M* elements from *X*
  - Sort them in main-memory
  - Write them at the end of file *Y*
- *Phase* 2: Repeat while there is more than one run in *Y:*
  - Empty *X*
  - *MergeAllRuns*(*Y*, *X*)
  - *X* is now called *Y*, *Y* is now called *X*

External-Memory Merging

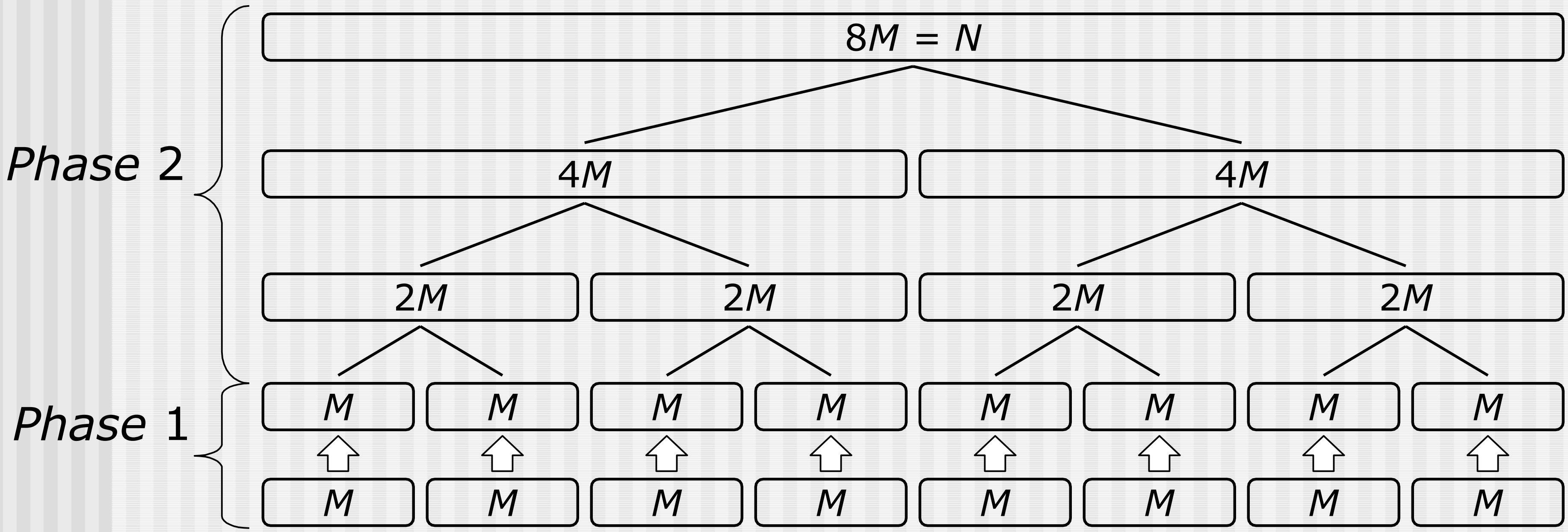
- *MergeAllRuns*(*Y*, *X*): repeat until the end of *Y*:
  - Call *TwowayMerge* to merge the next two runs from *Y* into one run, which is written at the end of *X*
- *TwowayMerge*: uses three main-memory arrays of size *B*



Analysis: Assumptions

- Assumptions and notation:
  - Disk page size:
    - B data elements
  - Data file size:
    - N elements, $n = N/B$ disk pages
  - Available main memory:
    - M elements, $m = M/B$ pages

Analysis



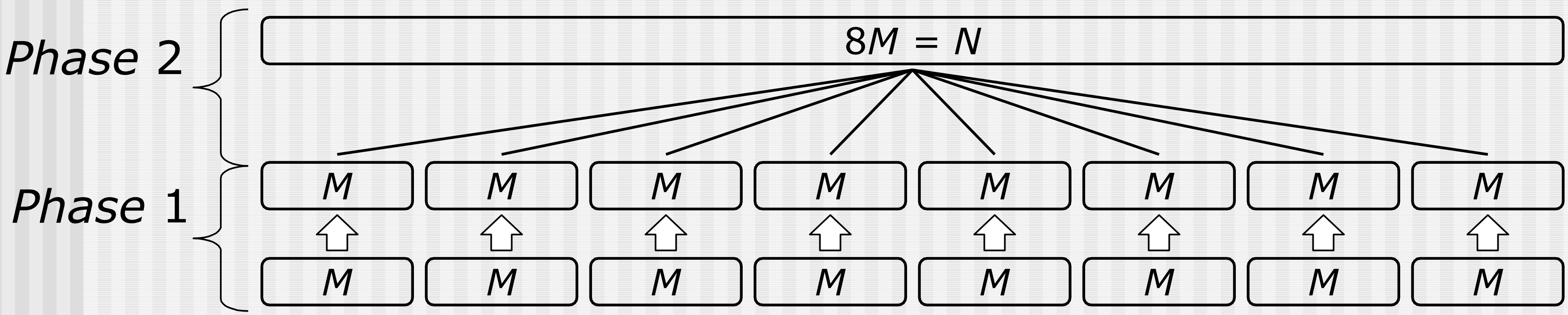
- Phase 1:
  - Read file X, write file Y: $2n = O(n)$ I/Os
- Phase 2:
  - One iteration: Read file Y, write file X: $2n = O(n)$ I/Os
  - Number of iterations: $\log_2 N/M = \log_2 n/m$

Analysis: Conclusions

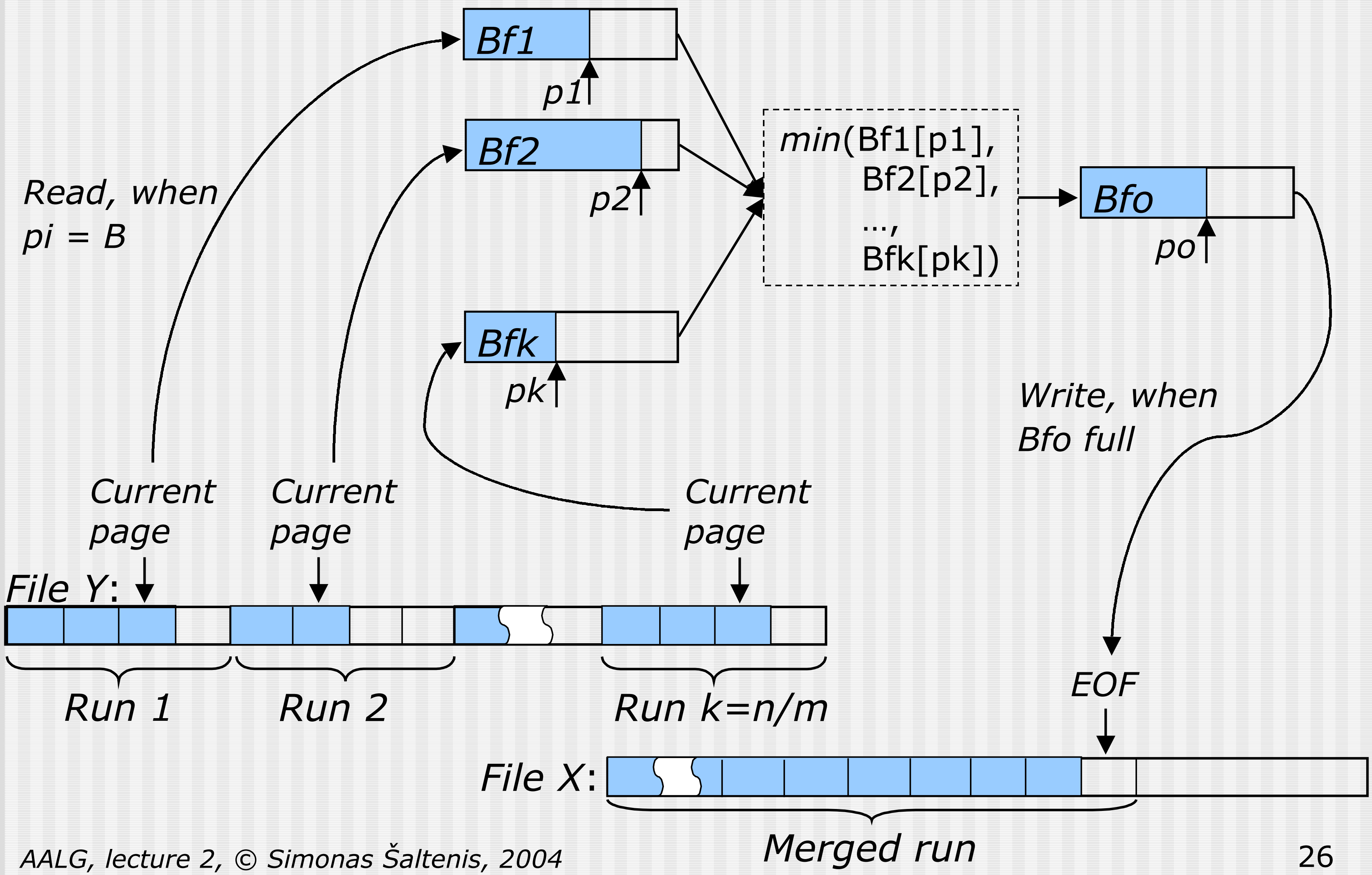
- Total running time of external-memory merge sort: $O(n \log_2 n/m)$
- We can do better!
- Observation:
  - Phase 1 uses all available memory
  - Phase 2 uses just 3 pages out of m available!!!

Two-Phase, Multiway Merge Sort

- Idea: merge all runs at once!
  - Phase 1: the same (do internal sorts)
  - Phase 2: perform *MultiwayMerge*(*Y*,*X*)



Multiway Merging



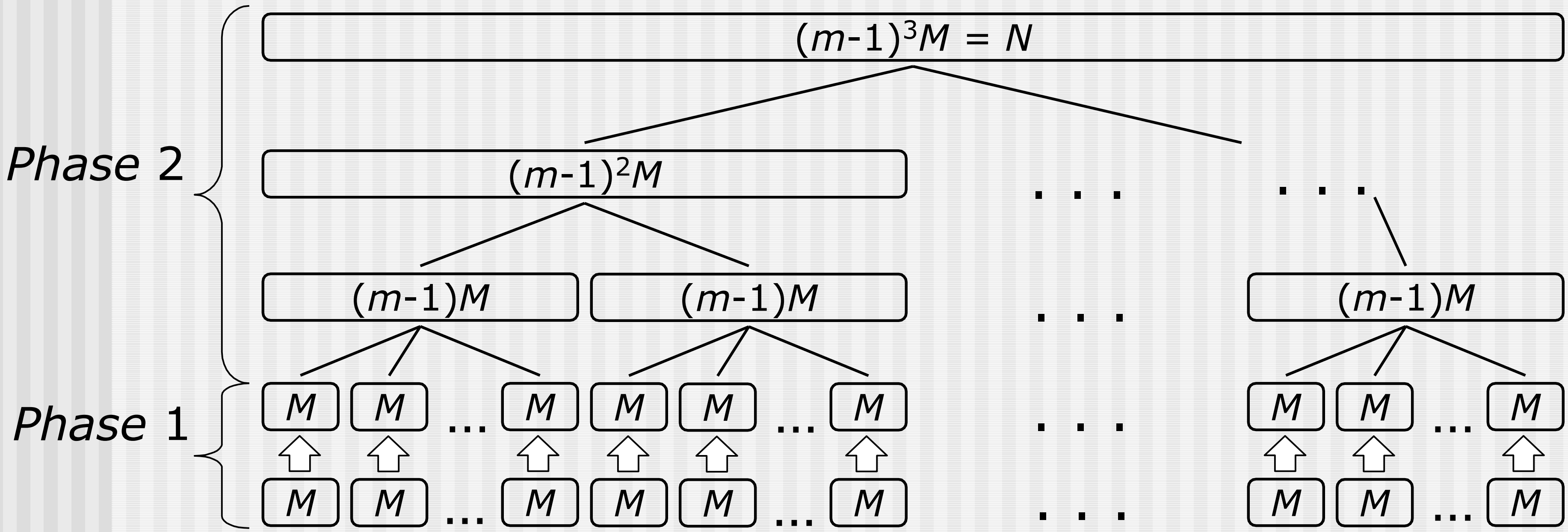
Analysis of TPMMS

- Phase 1: O(n), Phase 2: O(n)
- Total: O(n) I/Os!
- The catch: files only of "limited" size can be sorted ☹
  - Phase 2 can merge a maximum of m-1 runs.
  - Which means: $N/M < m-1$
  - *How large files can we sort with TPMMS on a machine with 128Mb main memory and disk page size of 16Kb?*

General Multiway Merge Sort

- What if a file is very large or memory is small?
- General *multiway merge sort*:
  - Phase 1: the same (do internal sorts)
  - Phase 2: do as many iterations of merging as necessary until only one run remains
    - Each iteration repeatedly calls *MultiwayMerge*(*Y*, *X*) to merge groups of *m-1* runs until the end of file *Y* is reached

Analysis



- Phase 1: O(n), each iteration of phase 2: O(n)
- How many iterations are there in phase 2?
  - Number of iterations: $\log_{m-1} N/M = \log_m n$
- Total running time: O($n \log_m n$) I/Os

Conclusions

- External sorting can be done in O(n log$_m$ n) I/O operations for any n
  - This is asymptotically optimal
- In practice, we can usually sort in O(n) I/Os
  - Use two-phase, multiway merge-sort