Advanced Algorithm Design and Analysis (Lecture 5)

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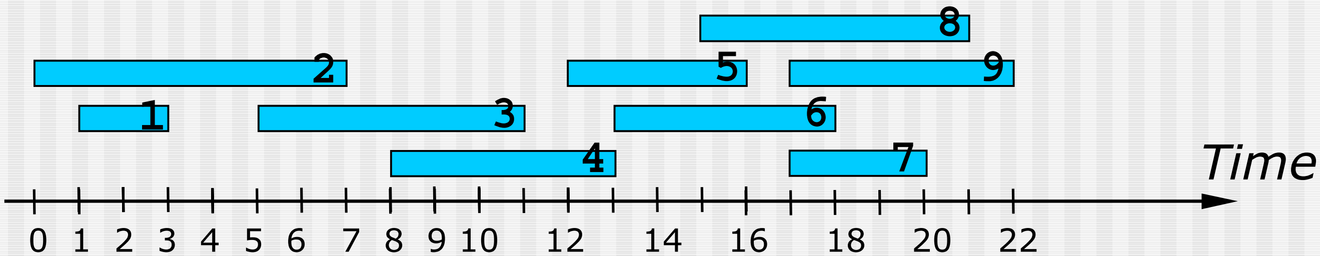
Greedy Algorithms

- Goals of the lecture:
  - *to understand the* ***principles*** *of the greedy algorithm design technique;*
  - *to understand the* ***example greedy algorithms*** *for activity selection and Huffman coding, to be able to* ***prove*** *that these algorithms find optimal solutions;*
  - *to be able to* ***apply*** *the greedy algorithm design technique.*

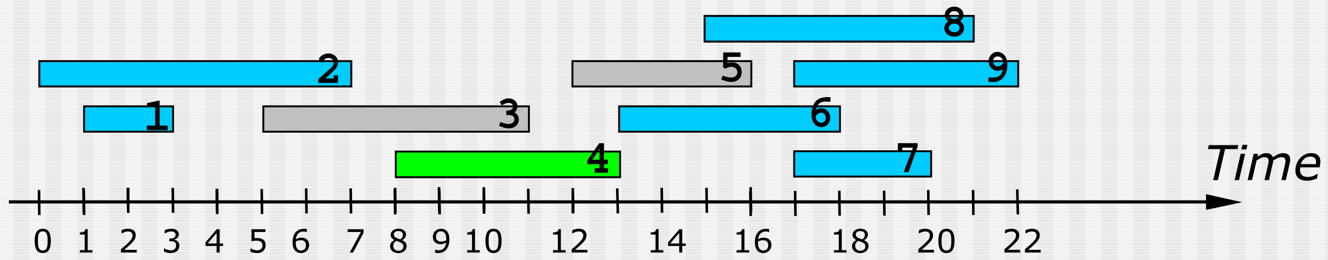
Activity-Selection Problem

- Input:
  - A set of n activities, each with start and end times: $A[i].s$ and $A[i].f$. The activity last during the period $[A[i].s, A[i].f)$
- Output:
  - The ***largest*** subset of mutually *compatible* activities
    - Activities are compatible if their intervals do not intersect



"Straight-forward" solution

- Let's just pick (schedule) one activity $A[k]$
  - This generates two set's of activities compatible with it: *Before*(k), *After*(k)
    - E.g., Before(4) = {1, 2}; After(4) = {6,7,8,9}



  - Solution:

$$MaxN(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \max_{1 \le k \le n} \{MaxN(Before(A)) + MaxN(After(A)) + 1\} & \text{if } A \ne \emptyset. \end{cases}$$

Dynamic Programming Alg.

- The recurrence results in a dynamic programming algorithm
  - Sort activities on the start or end time (for simplicity assume also "sentinel" activities *A*[0] and *A*[*n*+1])
  - Let S_{ij} – a set of activities after *A*[*i*] and before *A*[*j*] and compatible with *A*[*i*] and *A*[*j*].
  - Let's have a two-dimensional array, s.t., *c*[*i*, *j*] = *MaxN*(S_{ij}).

$$c[i,j] = \begin{cases} 0 & \text{if } S_{ij} = \varnothing, \\ \max\limits_{i<k<j}\{c[1,k] + c[k,j] + 1\} & \text{if } S_{ij} \neq \varnothing. \end{cases}$$

  - *MaxN*(*A*) = *MaxN*($S_{0,n+1}$) = *c*[0, *n*+1]

Dynamic Programming Alg. II

- Does it really work correctly?
  - We have to prove the optimal sub-structure:
    - *If an optimal solution A to S_{ij} includes A[k], then solutions to S_{ik} and S_{kj} (as parts of A) must be optimal as well*
    - To prove use "cut-and-paste" argument
- What is the running time of this algorithm?

Greedy choice

- What if we could choose "the best" activity (as of now) and be sure that it belongs to an optimal solution
  - We wouldn't have to check out all these sub-problems and consider all currently possible choices!
- Idea: Choose the activity that finishes first!
  - Then, solve the problem for the remaining compatible activities

```
MaxN(A[1..n], i)  //returns a set of activities
m ← i + 1
while m ≤ n and A[m].s < A[i].f do
   m ← m + 1
if m ≤ n then return {A[m]} ∪ MaxN(A, m)
         else return ∅
```

Greedy-choice property

- What is the running time of this algorithm?
- Does it find an optimal solution?:
  - We have to prove the *greedy-choice property*, i.e., that our locally optimal choice belongs to some globally optimal solution.
  - We have to prove the *optimal sub-structure* property (we did that already)
- The challenge is to choose the right interpretation of "the best choice":
  - How about the activity that starts first
    - Show a *counter-example*

Data Compression

- *Data compression* problem – strings *S* and *S′*:
  - *S* -> *S′* -> *S*, such that |*S′*|<|*S*|
- Text compression by coding with *variable-length* code:
  - Obvious idea – assign short codes to frequent characters: "**abracadabra**"

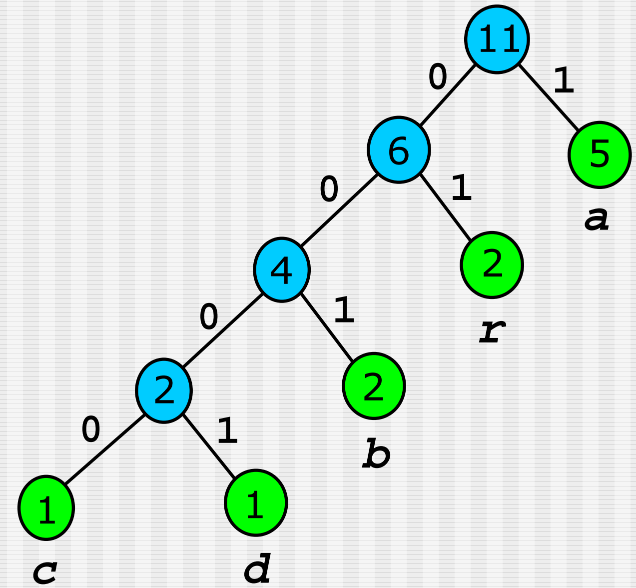
  Frequency table:

	a	**b**	**c**	**d**	**r**
Frequency	5	2	1	1	2
Fixed-length code	000	001	010	011	100
Variable-length code	1	001	0000	0001	01

  - How much do we save in this case?

Prefix code

- Optimal code for given frequencies:
  - Achieves the minimal length of the coded text
- *Prefix code*: no codeword is prefix of another
  - It can be shown that optimal coding can be done with prefix code



- We can store all codewords in a *binary trie* – very easy to decode
  - Coded characters in leaves
  - Each node contains the sum of the frequencies of all descendants

Optimal Code/Trie

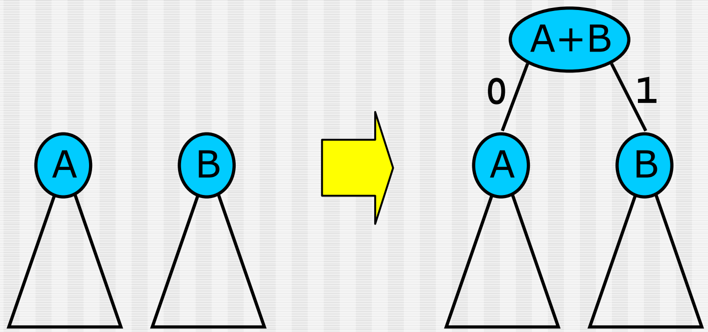
- The *cost* of the coding trie *T*:

$$B(T) = \sum_{c \in C} f(c) d_T(c)$$

  - *C* – the alphabet,
  - *f*(*c*) – frequency of character *c*,
  - $d_T(c)$ – depth of *c* in the trie (length of code in bits)
- Optimal trie – the one that minimizes *B*(*T*)
- Observation – optimal trie is always full:
  - Every non-leaf node has two children. Why?

Huffman Algorithm - Idea

- Huffman algorithm, builds the code trie bottom up. Consider a forest of trees:
  - Initially – one separate node for each character.
  - In each step – join two trees into a larger tree



  - Repeat this until one tree (trie) remains.
  - Which trees to join? Greedy choice – the trees with the **smallest** frequencies!

Huffman Algorithm

```
Huffman(C)
Q.build(C) // Builds a min-priority queue on frequences
for i ← 1 to n-1 do
   Allocate new node z
   x ← Q.extractMin()
   y ← Q.extractMin()
   z.setLeft(x)
   z.setRight(y)
   z.setF(x.f() + y.f())
   Q.insert(z)
return Q.extractMin() // Return the root of the trie
```

- What is its running time?
- Run the algorithm on: "**oho ho, Ole**"

Correctness of Huffman

- Greedy choice property:
  - Let *x*, *y* – two characters with lowest frequencies. Then there exists an optimal prefix code where codewords for *x* and *y* have the same length and differ only in the last bit
  - Let's prove it:
    - Transform an optimal trie *T* into one (*T''*), where *x* and *y* are max-depth siblings. Compare the costs.

Correctness of Huffman

- Optimal sub-structure property:
  - Let x, y – characters with minimum frequency
  - $C' = C - \{x,y\} \cup \{z\}$, such that $f(z) = f(x) + f(y)$
  - Let T' be an optimal code trie for C'
  - Replace leaf z in T' with internal node with two children x and y
  - The result tree T is an optimal code trie for C
- Proof a little bit more involved than a simple "cut-and-paste" argument

Elements of Greedy Algorithms

- Greedy algorithms are used for optimization problems
  - A number of choices have to be made to arrive at an optimal solution
  - At each step, make the “locally best” choice, without considering all possible choices and solutions to sub-problems induced by these choices (compare to dynamic programming)
  - After the choice, only one sub-problem remains (smaller than the original)
- Greedy algorithms usually sort or use priority queues

Elements of Greedy Algorithms

- First, one has to prove the *optimal sub-structure* property
  - the simple "cut-and-paste" argument may work
- The main challenge is to decide the interpretation of "the best" so that it leads to a global optimal solution, i.e., you can prove the *greedy choice property*
  - The proof is usually constructive: takes a hypothetical optimal solution without the specific greedy choice and transforms into one that has this greedy choice.
  - Or you find counter-examples demonstrating that your greedy choice does not lead to a global optimal solution.

Other Greedy Algorithms

- Find a minimum spanning tree in a weighted graph
- Coin changing