Inference in Bayesian networks using the junction tree approach

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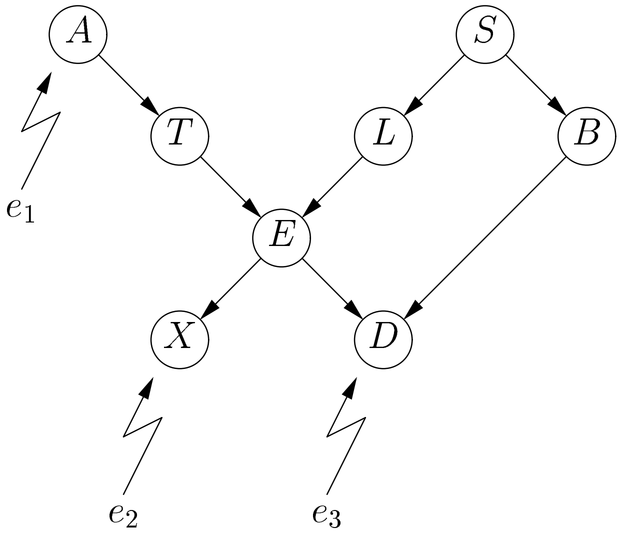
The lecture is to a large extent based on Chapters 4 and 5 in

Finn V. Jensen (1996). *An introduction to Bayesian Networks*, UCL Press Ltd., London

Making junction trees and propagating in them

Motivation

$$P(L\,|\,\mathbf{e}) \propto \sum_E \sum_B \left[\sum_T P(A=a)P(T\,|\,A=a)P(E\,|\,T,L)\right] P(D=d\,|\,B,E)P(X=x\,|\,E)\sum_S P(S)P(L\,|\,S)P(B\,|\,S)$$



$$P(T,e_1) = P(A=a)P(T\,|\,A=a)$$

$$P(B,L) = \sum_S P(S)P(L\,|\,S)P(B\,|\,S)$$

$$P(E,e_1\,|\,L) = \sum_T P(T,e_1)P(E\,|\,T,L)$$

$$P(B,L,E,\mathbf{e}) = P(E,e_1\,|\,L)P(e_2,e_3\,|\,B,E)P(B,L)$$

$$P(e_2,e_3\,|\,B,E) = P(D=d\,|\,B,E)P(e_2\,|\,E)$$

$$P(e_2\,|\,E) = P(X=x\,|\,E)$$

$$P(L\,|\,\mathbf{e}) = \frac{\sum_{B,E} P(B,L,E,\mathbf{e})}{\sum_{B,L,E} P(B,L,E,\mathbf{e})}$$

Making junction trees and propagating in them

Motivation

$$P(T, e_1) = P(A=a)P(T \,|\, A=a)$$

$$P(B, L) = \sum_S P(S)P(L \,|\, S)P(B \,|\, S)$$

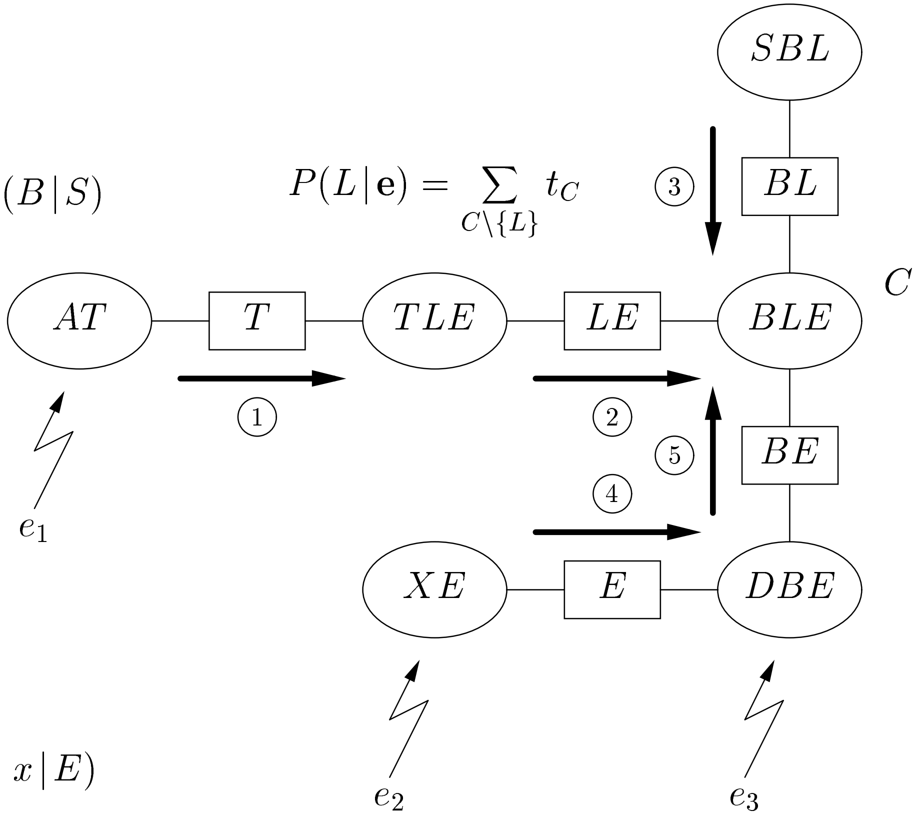
$$P(E, e_1 \,|\, L) = \sum_T P(T, e_1)P(E \,|\, T, L)$$

$$P(B, L, E, \mathbf{e}) = P(E, e_1 \,|\, L)P(e_2, e_3 \,|\, B, E)P(B, L)$$

$$P(e_2, e_3 \,|\, B, E) = P(D=d \,|\, B, E)P(e_2 \,|\, E)$$

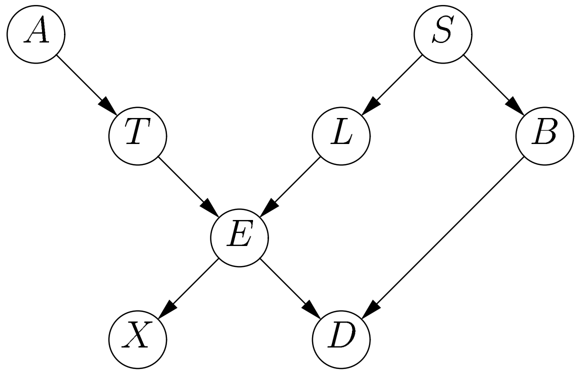
$$P(e_2 \,|\, E) = P(X=x \,|\, E)$$

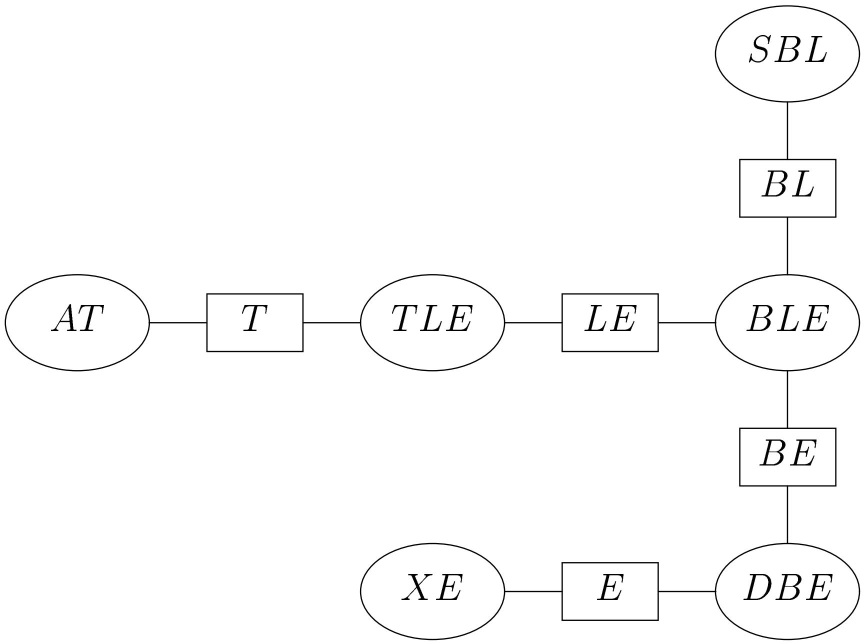
$$P(L \,|\, \mathbf{e}) = \sum_{C \setminus \{L\}} t_C$$



Making junction trees and propagating in them

Motivation





1. How do we perform inference in a junction tree?
2. How do we build a junction tree?

Making junction trees and propagating in them

Outline

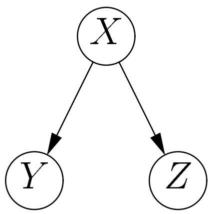
- Propagation in junction trees
  - Initialisation of a junction tree
    - Conditional probabilities become potentials
    - Joint probability re-expressed through potentials
  - Message absorption
    - (Local) consistence enforced
    - Invariance of t_U
  - Junction tree property
    - Local consistence $\Longrightarrow$ global consistence
  - Message passing in junction trees
    - Message scheduling
    - Global consistence achieved by passing at most $2(n-1)$ messages
- Construction of junction trees
  - Singly connected DAGs
  - Multiply connected DAGs
  - From DAG to junction tree
    - Moralisation
    - Triangulation
    - From triangulated graph to junction tree
  - Chain graphs
- Advanced inference in junction trees
  - Computing arbitrary marginals
  - Conflict analysis
  - Sensitivity analysis
  - Value of information

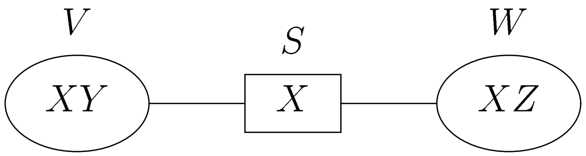
Propagation in junction trees

Notation

$\mathcal{C}$: set of all clusters of a junction tree
$\mathcal{S}$: set of all separators of a junction tree
t_A : table over $\mathrm{Sp}(A)$

Example





$\mathcal{C} = \{V, W\}$, $\mathcal{S} = \{S\}$
t_V: table over $\mathrm{Sp}(V) = \mathrm{Sp}(\{X, Y\}) = \mathrm{Sp}(X) \times \mathrm{Sp}(Y) = \{(x_1, y_1), (x_1, y_2), \ldots, (x_n, y_m)\}$
t_S: table over $\mathrm{Sp}(S) = \mathrm{Sp}(X) = \{x_1, \ldots, x_n\}$
t_W: table over $\mathrm{Sp}(W) = \mathrm{Sp}(\{X, Z\}) = \mathrm{Sp}(X) \times \mathrm{Sp}(Z) = \{(x_1, z_1), (x_1, z_2), \ldots, (x_n, z_r)\}$

To each cluster, C, (and each separator, S) of a junction tree is associated a table over the product space of the spaces of the variables of C (S).

Propagation in junction trees

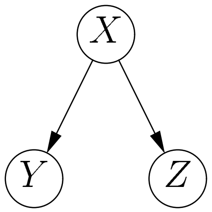
Initialisation

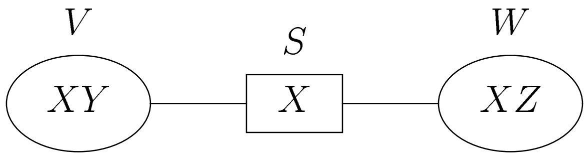
The cluster tables and separator tables of a junction tree are initialised as follows.

1. For each $C \in \mathcal{C}$ let $t_C = \mathbf{1}$
2. For each $S \in \mathcal{S}$ let $t_S = \mathbf{1}$
3. For each variable $A \in U$ choose exactly one cluster, C, containing $\mathrm{pa}(A) \cup \{A\}$ and let

$$t_C := t_C * P(A \,|\, \mathrm{pa}(A))$$

Example





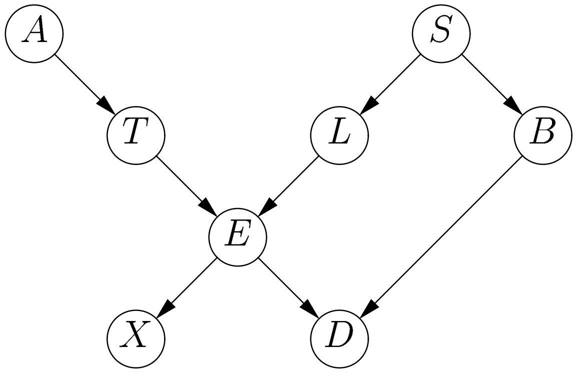
$$t_V = P(X)P(Y\,|\,X) \qquad t_S = \mathbf{1} \qquad t_W = P(Z\,|\,X)$$

$$P(U) = P(X)P(Y\,|\,X)P(Z\,|\,X)$$

$$t_U = \frac{t_V \cdot t_W}{t_S} = P(X)P(Y\,|\,X)P(Z\,|\,X) = P(U)$$

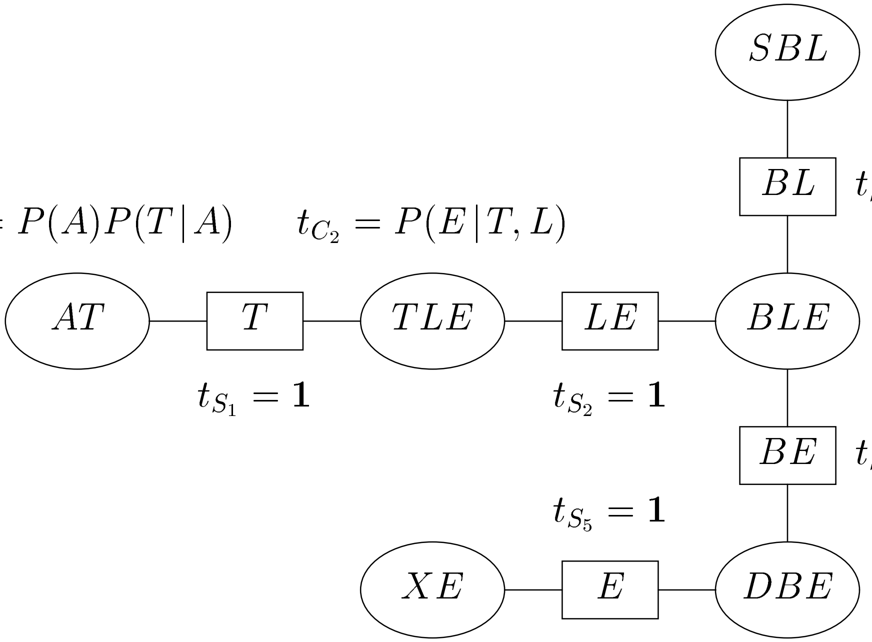
Propagation in junction trees

Initialisation, Asia



$t_{C_3} = P(S)P(L\,|\,S)P(B\,|\,S)$

$t_{C_1} = P(A)P(T\,|\,A)$ $\quad t_{C_2} = P(E\,|\,T, L)$



$t_{S_3} = \mathbf{1}$

$t_{C_4} = \mathbf{1}$

$t_{S_1} = \mathbf{1}$ $\quad t_{S_2} = \mathbf{1}$

$t_{S_4} = \mathbf{1}$

$t_{S_5} = \mathbf{1}$

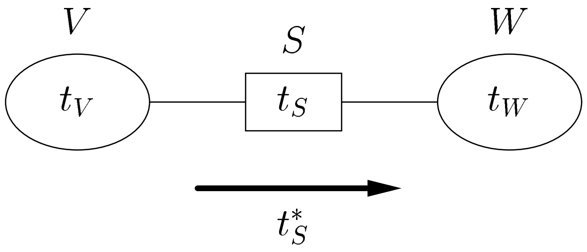
$t_{C_6} = P(X\,|\,E)$ $\quad t_{C_5} = P(D\,|\,B, E)$

$$\begin{aligned} P(U) \;&=\; P(A)P(T\,|\,A)P(S)P(L\,|\,S)P(B\,|\,S) \\ &\;\; P(E\,|\,T, L)P(D\,|\,B, E)P(X\,|\,E) \end{aligned}$$

$$t_U = \frac{\prod_{i=1}^{6} t_{C_i}}{\prod_{i=1}^{5} t_{S_i}} = \prod_{i=1}^{6} t_{C_i} = P(U)$$

Propagation in junction trees

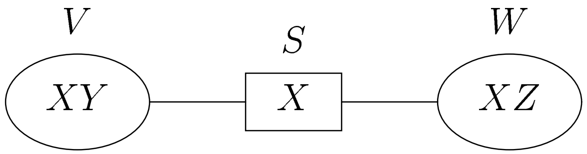
Message absorption



1. $t_S^* = \sum_{V \setminus S} t_V$ (compute message)
2. $t_S := t_S^*$ (store message in S)
3. $t_W := t_W \dfrac{t_S^*}{t_S}$ (W absorbs message)

Propagation in junction trees

Message absorption, example



$$t_V = P(X)P(Y \mid X) \qquad t_S = \mathbf{1} \qquad t_W = P(Z \mid X)$$

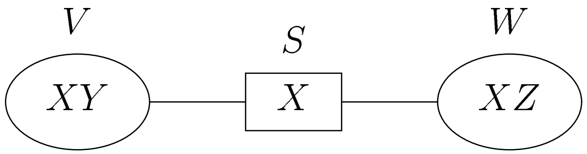
$$\xrightarrow[t_S^*]{\qquad\qquad}$$

1. $t_S^* = \sum_Y t_V = \sum_Y P(X,Y) = P(X)$
2. $t_S := t_S^* = P(X)$
3. $t_W := t_W \dfrac{t_S^*}{t_S} = P(Z \mid X)\dfrac{P(X)}{\mathbf{1}} = P(X,Z)$

NB: Now t_V, t_S, t_W contain true marginals!

Propagation in junction trees

Message absorption, example



$$t_V = P(X)P(Y \,|\, X) \qquad t_S = \mathbf{1} \qquad t_W = P(Z \,|\, X)$$

$$\xleftarrow[\quad t_S^* \quad]{}$$

1. $t_S^* = \sum_Z t_W = \sum_Z P(Z \,|\, X) = \mathbf{1}$
2. $t_S := t_S^* = \mathbf{1}$ $\qquad$ (NB: $t_S^* = t_S$)
3. $t_V := t_V \dfrac{t_S^*}{t_S} = t_V \dfrac{\mathbf{1}}{\mathbf{1}} = t_V$

NB: Nothing changed!

Propagation in junction trees

Message absorption, consistence

The link between V and W is *consistent* if

$$\sum_{V\setminus S} t_V = t_S = \sum_{W\setminus S} t_W.$$

Then absorption doesn't change anything. If all links of a junction tree, T, are consistent, then T is consistent.

Assume that the link is consistent, but now some evidence changes t_V to t_V^*. Then after W has absorbed from V, the three tables all contain V's new information on S:

$$\sum_{W\setminus S} t_W^* = \sum_{W\setminus S} t_W \frac{t_S^*}{t_S} = \frac{t_S^*}{t_S} \sum_{W\setminus S} t_W = \frac{t_S^*}{t_S} t_S = t_S^* = \sum_{V\setminus S} t_V^*.$$

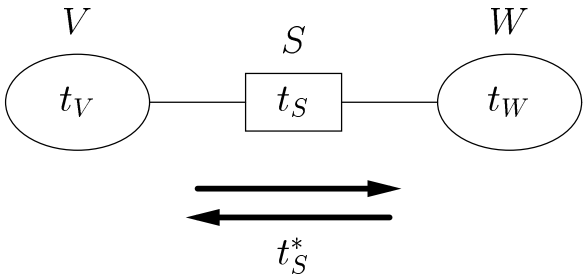
Or, shorter

$$\sum_{W\setminus S} t_W^* = t_S^* = \sum_{V\setminus S} t_V^*,$$

i.e., absorption makes the link consistent.

Propagation in junction trees

Message absorption, supportiveness



W can only absorb from V (and vice versa) if

$$t_S^*(x) > 0 \Rightarrow t_S(x) > 0 \ \text{ for all } x \in \mathrm{Sp}(S).$$

If this requirement is fulfilled, the link is *supportive*. Supportiveness is preserved under absorption, since

$$t_W^* = t_W \frac{t_S^*}{t_S}, \quad \text{where} \quad t_S^* = \sum_{V \setminus S} t_V.$$

That is, any zero entry in t_S^* is also a zero entry in t_W^*, or, more precisely,

$$t_S^*(x_S) = 0 \Rightarrow t_W^*(x_W) = 0$$

for each $x_W \in \mathrm{Sp}(W)$ for which $x_S \in \mathrm{Sp}(S)$ is a marginal entry of x_W with respect to S (i.e., $x_S = (x_W)_S$).

Propagation in junction trees

Message absorption, invariance of t_U

$t_U = \dfrac{\prod_{C\in\mathcal{C}} t_C}{\prod_{S\in\mathcal{S}} t_S}$ is invariant under absorption, since

$$t_U^* = \frac{\prod_{C\in\mathcal{C}} t_C^*}{\prod_{S\in\mathcal{S}} t_S^*} = \frac{\prod_{C\in\mathcal{C}\setminus\{W\}} t_C}{\prod_{S\in\mathcal{S}\setminus\{S'\}} t_S} \cdot \frac{t_W^*}{t_{S'}^*} = \frac{\prod_{C\in\mathcal{C}\setminus\{W\}} t_C}{\prod_{S\in\mathcal{S}\setminus\{S'\}} t_S} \cdot \frac{t_W \frac{t_{S'}^*}{t_{S'}}}{t_{S'}^*} = \frac{\prod_{C\in\mathcal{C}\setminus\{W\}} t_C}{\prod_{S\in\mathcal{S}\setminus\{S'\}} t_S} \cdot \frac{t_W}{t_{S'}} = \frac{\prod_{C\in\mathcal{C}} t_C}{\prod_{S\in\mathcal{S}} t_S} = t_U,$$

when W absorbs from V via S'.

In essence,

$$\frac{t_W^*}{t_{S'}^*} = \frac{t_W \frac{t_{S'}^*}{t_{S'}}}{t_{S'}^*} = \frac{t_W}{t_{S'}}.$$

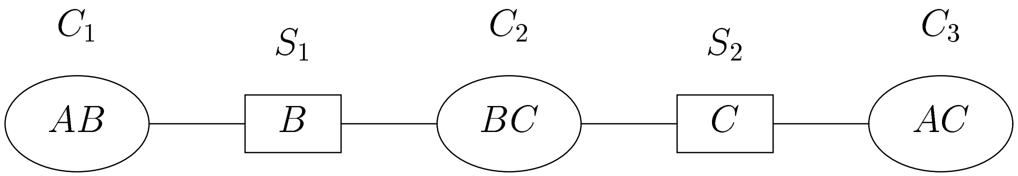
Propagation in junction trees

Junction tree property, motivation

If T is a consistent cluster tree, then if $A \in V$ and $A \in W$,

$$\sum_{V \setminus \{A\}} t_V = \sum_{W \setminus \{A\}} t_W \qquad (*)$$

if V and W are neighbours in T. If T is given by



there is, however, no guarantee that $(*)$ is fulfilled for C_1, C_3! For example, information about A in C_1 cannot be communicated to C_3!

Therefore we need *global consistence*, i.e.,

$$\sum_{V \setminus I} t_V = \sum_{W \setminus I} t_W, \qquad I = V \cap W$$

for *any* pair of clusters V, W.

Propagation in junction trees

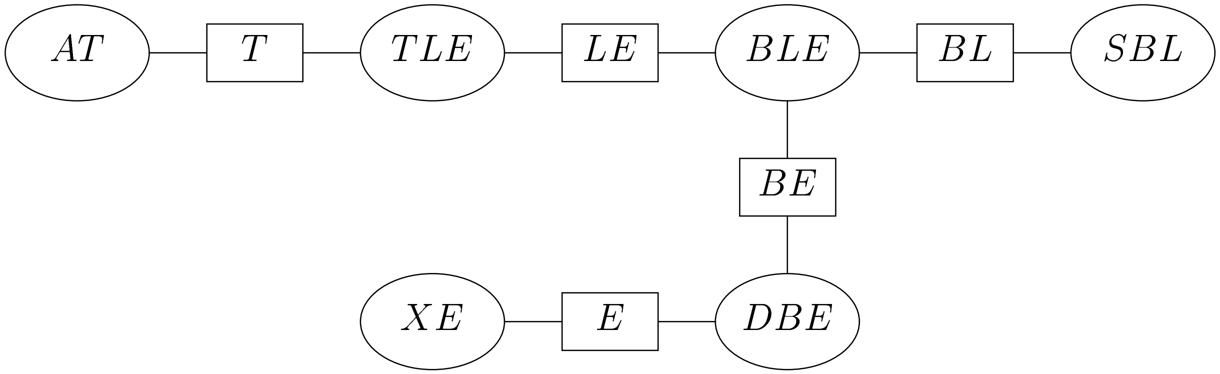
Junction tree property, definition

To guarantee global consistence, we must require the *junction tree property* to hold:

> A cluster tree is a *junction tree* if for each pair of clusters V, W, all clusters on the path between V and W contain $V \cap W$.

$$\boxed{\textit{A consistent junction tree is globally consistent}}$$

Example

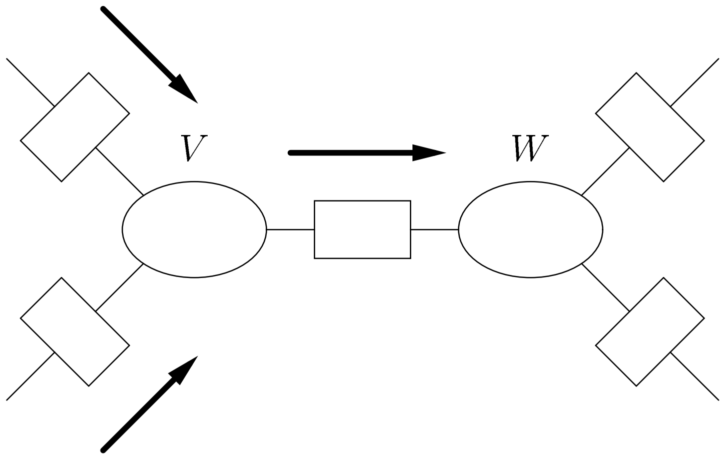


Propagation in junction trees

Message passing

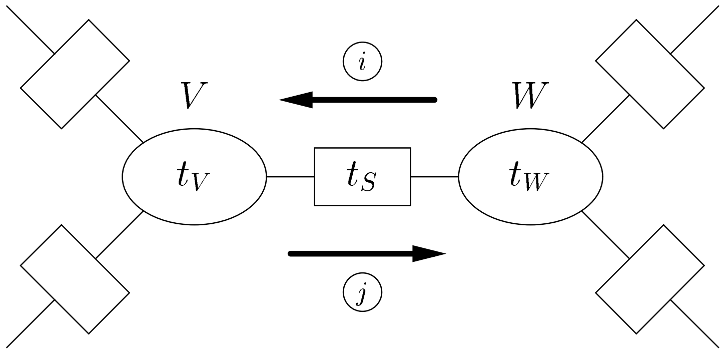
"W absorbs a message from V" $\equiv$ "V sends (or passes) a message to W"

V can send exactly one message to W, but only when it has received a message from each of its other neighbours



Propagation in junction trees

Message passing, consistence



$i < j$

Before message i: $t_V \quad t_S \quad t_W^*$ (W can have received messages from other neighbours)

After message i: $t_V^* \quad t_S^* \quad t_W^*$ $\left(t_S^* = \sum_{W \setminus S} t_W^*, \quad t_V^* = t_V \frac{t_S^*}{t_S}\right)$

Before message j: $t_V^{**} \quad t_S^* \quad t_W^*$ (W hasn't reveived further messages)

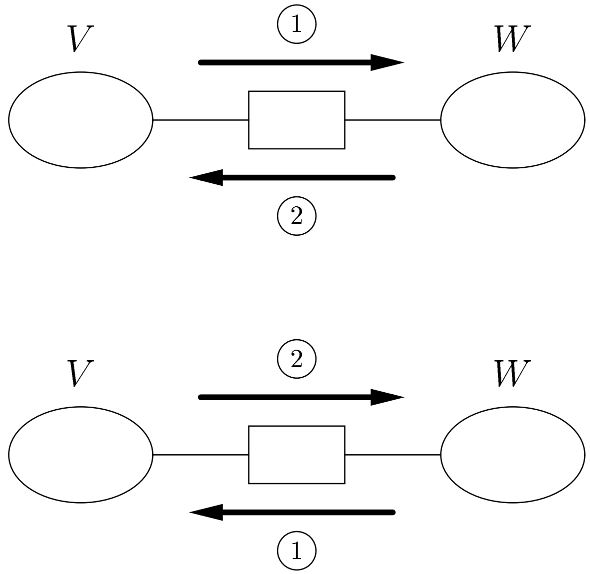
After message j: $t_V^{**} \quad t_S^{**} \quad t_W^{**}$ $\left(t_S^{**} = \sum_{V \setminus S} t_V^{**}, \quad t_W^{**} = t_W \frac{t_S^{**}}{t_S^*}\right)$

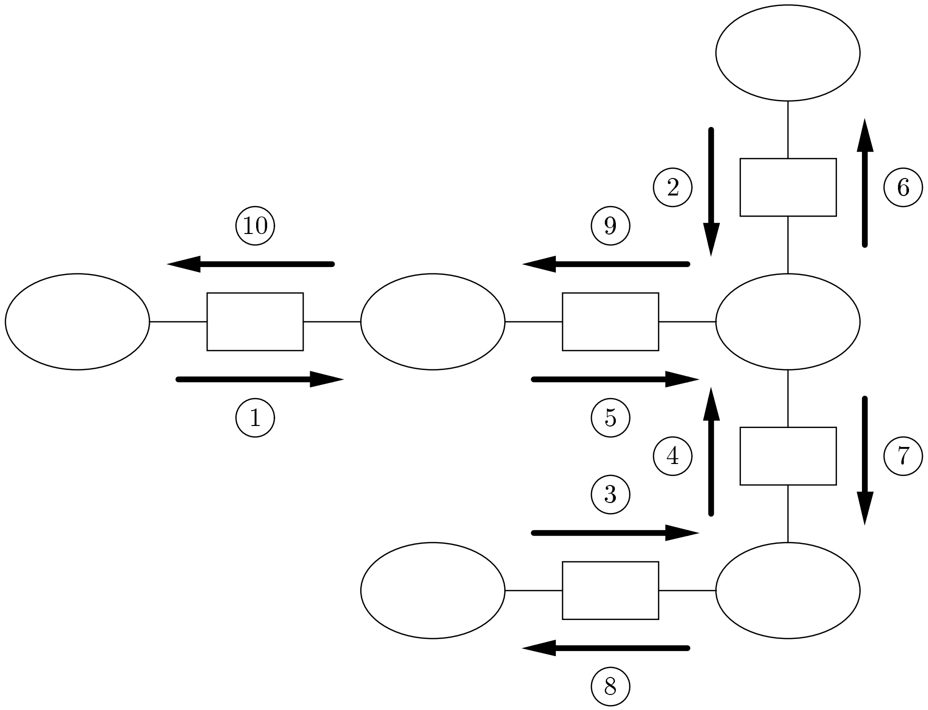
When both messages have been sent, the link is consistent:

$$\sum_{W \setminus S} t_W^{**} = \sum_{W \setminus S} t_W^* \frac{t_S^{**}}{t_S^*} = \frac{t_S^{**}}{t_S^*} \sum_{W \setminus S} t_W^* = \frac{t_S^{**}}{t_S^*} t_S^* = t_S^{**} = \sum_{V \setminus S} t_V^{**}$$

Propagation in junction trees

Message scheduling, examples



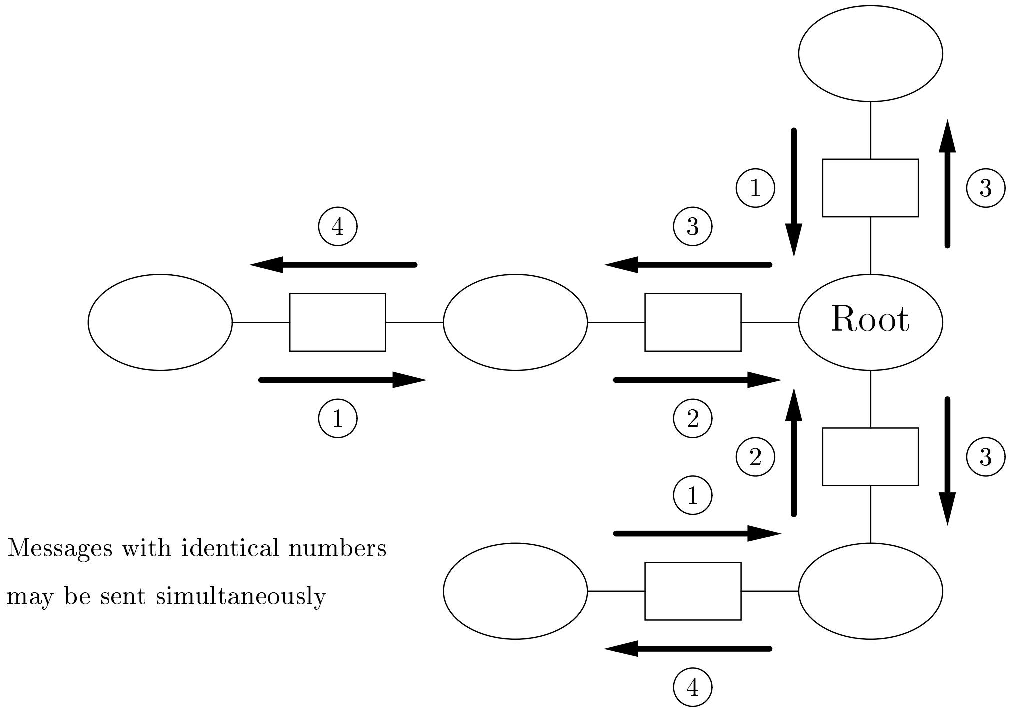


NB: When a full schedule has been executed, all links are consistent (i.e., the tree is consistent).

Propagation in junction trees

Inward/outward message passing

Performing an inward pass (from the leaves towards a root cluster) followed by an outward pass (from the root cluster towards the leaves) provides an easy-to-implement message scheduling algorithm.



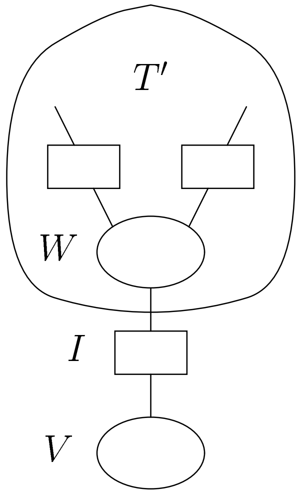
Propagation in junction trees

$\mathbf{t_V = \sum_{U \setminus V} t_U}$ in a consistent junction tree

Let T be a consistent junction tree over variables U, and let $t_U = \dfrac{\prod_{C \in \mathcal{C}} t_C}{\prod_{S \in \mathcal{S}} t_S}$. Then $t_V = \sum_{U \setminus V} t_U$, $V \in \mathcal{C}$.

Proof: Induction on the number of clusters, n. Holds for $n = 1$. Assume it holds for $n \leq k$, and let $n = k + 1$ and $U' = U \setminus (V \setminus I)$ (see figure). Then

$$t_U = \frac{\prod_{C \in \mathcal{C} \setminus \{V\}} t_C}{\prod_{S \in \mathcal{S} \setminus \{I\}} t_S} \cdot \frac{t_V}{t_I} = t_{U'} \cdot \frac{t_V}{t_I}.$$



Consistence of T gives

$$\sum_{V \setminus I} t_V = t_I = \sum_{W \setminus I} t_W,$$

hence

$$\sum_{V \setminus I} t_U = \sum_{V \setminus I} t_{U'} \cdot \frac{t_V}{t_I} = t_{U'} \cdot \frac{\sum_{V \setminus I} t_V}{t_I} = t_{U'}.$$

Therefore, by the induction hypothesis, for all C in T'

$$\sum_{U \setminus C} t_U = \sum_{U' \setminus C} \sum_{V \setminus I} t_U = \sum_{U' \setminus C} t_{U'} = t_C \qquad \text{and} \qquad \sum_{U \setminus V} t_U = \sum_{U' \setminus I} t_{U'} \cdot \frac{t_V}{t_I} = \frac{t_V}{t_I} \sum_{U' \setminus I} t_{U'} = \frac{t_V}{t_I} t_I = t_V. \qquad \square$$

Propagation in junction trees

A consistent junction tree contains true marginals

Let T be a consistent junction tree over variables U, and let $P(U)$ be the joint probability distribution for U given by the corresponding Bayesian network. Then for each $C \in \mathcal{C}$

$$t_C = \sum_{U \setminus C} P(U) = P(C).$$

Proof: Before propagation we have

$$t_U = \frac{\prod_{C \in \mathcal{C}} t_C}{\prod_{S \in \mathcal{S}} t_S} = \prod_{C \in \mathcal{C}} t_C = P(U).$$

Since t_U is invariant under message passing, i.e.,

$$t_U = t_U^* = \frac{\prod_{C \in \mathcal{C}} t_C^*}{\prod_{S \in \mathcal{S}} t_S^*} = P(U)$$

after propagation (execution of a full message schedule), and $t_C = \sum_{U \setminus C} t_U$, the result follows. $\square$

Propagation in junction trees

... also when there's evidence

$$P(U, \mathbf{e}) = \prod_{A \in U} P(A \,|\, \mathrm{pa}(A)) * \mathcal{E}_A = \prod_{A \in U} P^*(A \,|\, \mathrm{pa}(A)),$$

where

$$\mathcal{E}_A(a) = \begin{cases} 1 & \text{if } A = a \text{ or if there is no evidence on } A \\ 0 & \text{otherwise.} \end{cases}$$

Before propagation,

$$t_U = \frac{\prod_{C \in \mathcal{C}} t_C}{\prod_{S \in \mathcal{S}} t_S} = \prod_{C \in \mathcal{C}} t_C = \prod_{A \in U} P^*(A \,|\, \mathrm{pa}(A)) = P(U, \mathbf{e}).$$

After propagation,

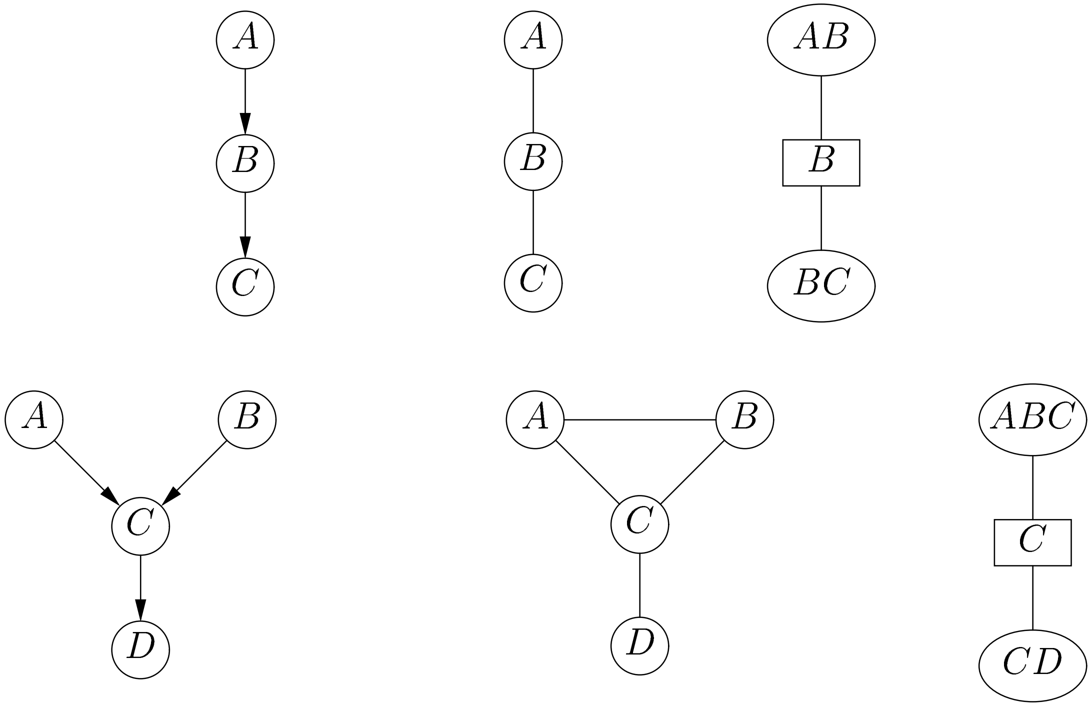
$$t_C = \sum_{U \setminus C} P(U, \mathbf{e}) = P(C, \mathbf{e}),$$

using the preceding result.

Construction of junction trees

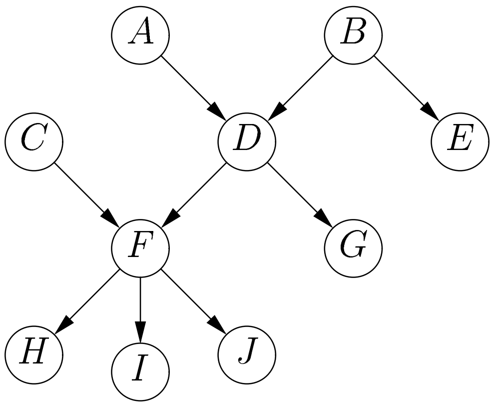
Singly connected DAGs

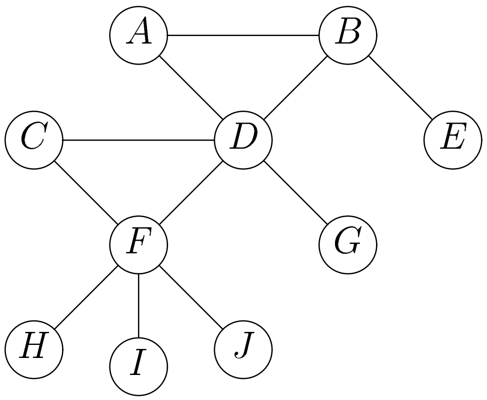
Remember: For each variable, A, there must be a cluster containing $\{A\} \cup \mathrm{pa}(A)$.

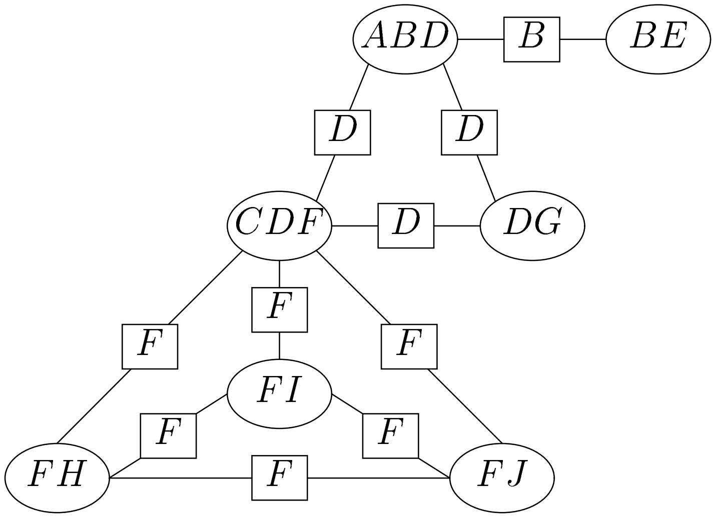


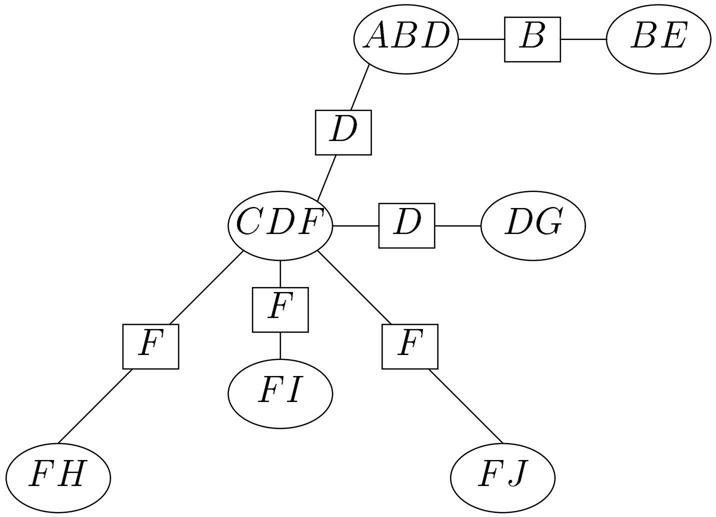
Construction of junction trees

Singly connected DAGs



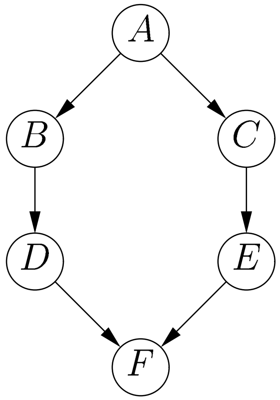




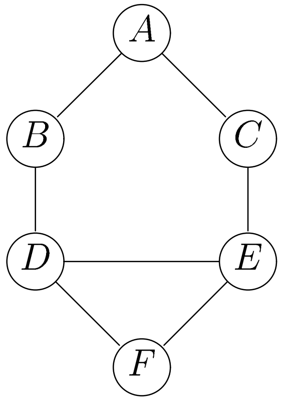


Construction of junction trees

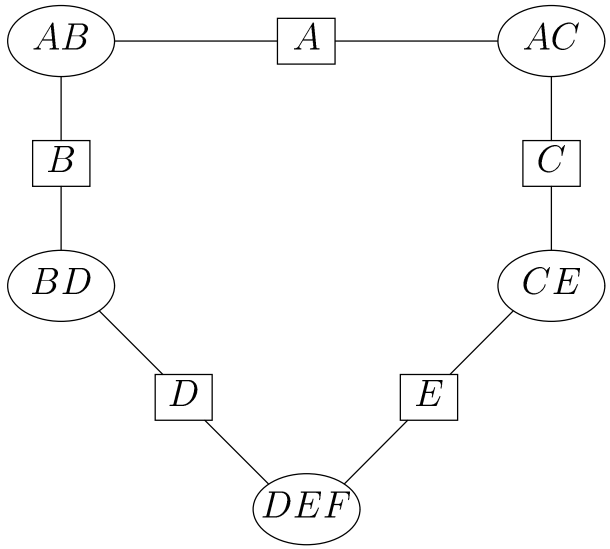
Multiply connected DAGs



DAG



Moral graph



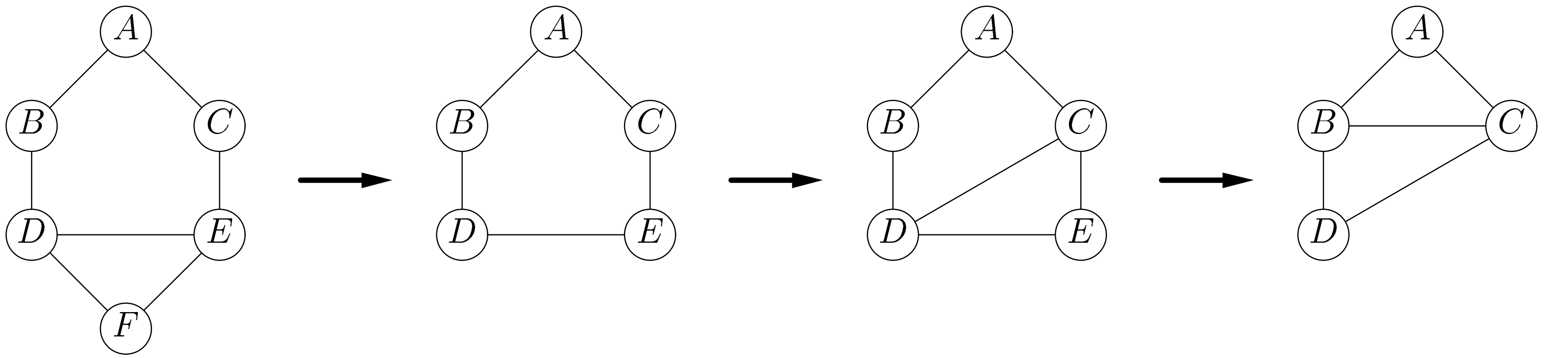
Junction graph

Chordless cycles of length 4 or greater in the moral graph yield indissoluble cycles in the junction graph

Construction of junction trees

Elimination of chordless cycles

All cycles of length 4 or greater including F has chord D—E $\Longrightarrow$ Dissolution of chordless cycles of length 4 or greater can be performed in the subgraph obtained by removing F

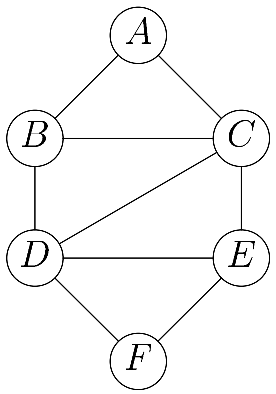


Eliminate F | Add C—D | Elim. E and add B—C | No chordless cycles!

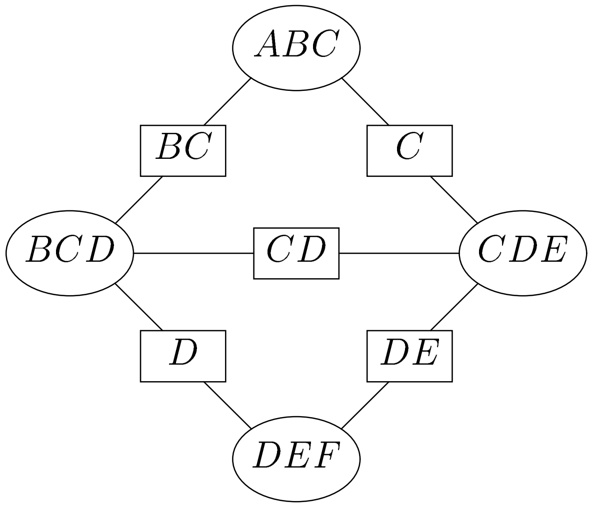
> An undirected graph without chordless cycles of length 4 or greater is called a *triangulated* graph

Construction of junction trees

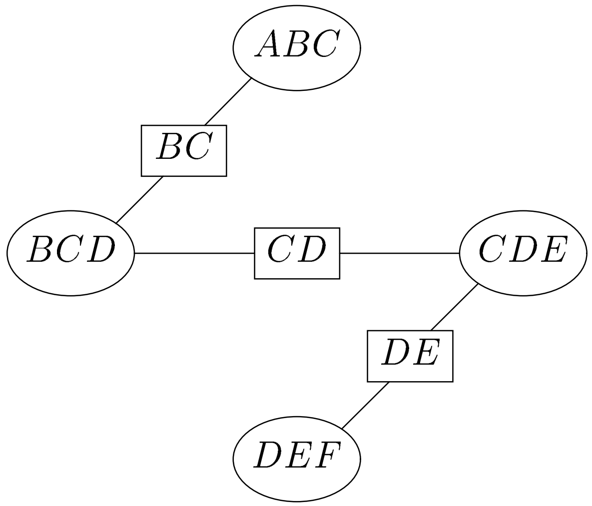
Multiply connected DAGs



Triangulated graph
Elim. seq.: F, E, D, C, B, A



Junction graph

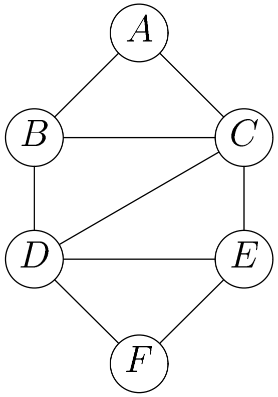


Junction tree

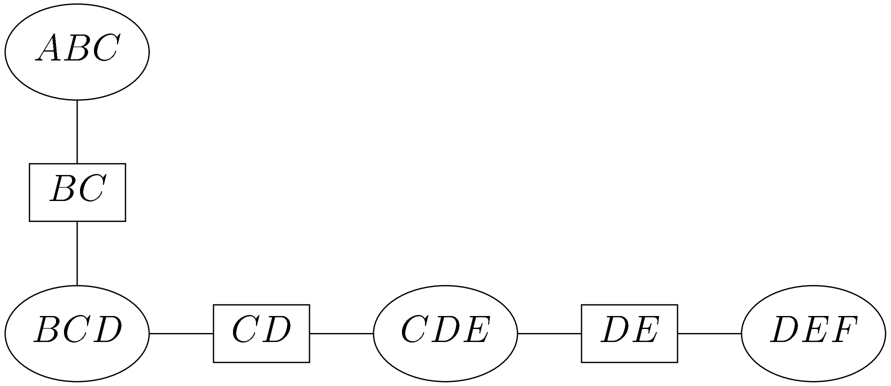
An undirected graph is triangulated if and only if its junction graph has a junction tree

Construction of junction trees

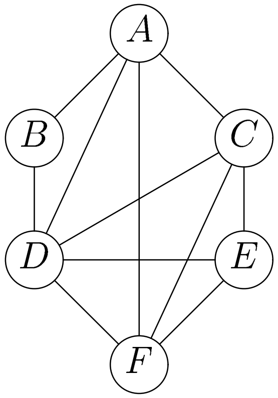
Finding a (near-)optimal elimination sequence is important



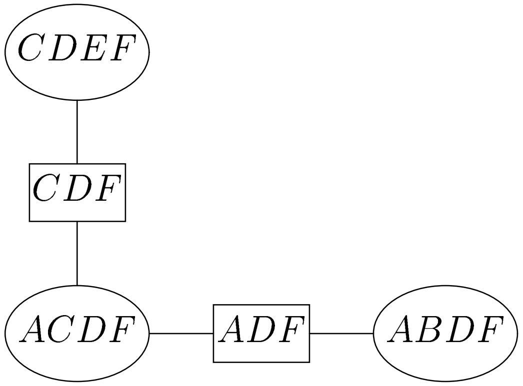
Elim. seq.: F, E, D, C, B, A



Cost: $4 \times 5^3 = 500$ with $|\mathrm{Sp}(A)| = \cdots = |\mathrm{Sp}(F)| = 5$



Elim. seq.: E, C, A, B, D, F

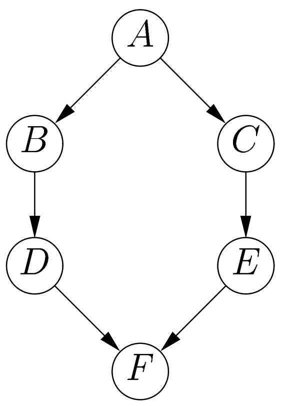


Cost: $3 \times 5^4 = 1875$ with $|\mathrm{Sp}(A)| = \cdots = |\mathrm{Sp}(F)| = 5$

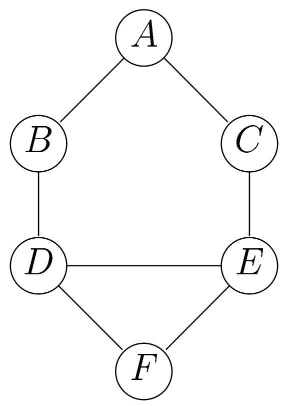
Construction of junction trees

From DAG to junction tree

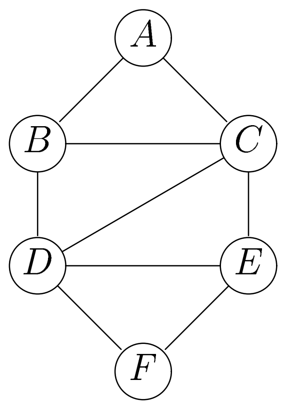
1. Moralise DAG
2. Triangulate moral graph (find good elimination sequence)
3. Identify the cliques (maximal complete subgraphs) of the triangulated graph
4. Create successively a link between two cliques whose intersection is maximal, and such that no cycles are created (Kruskal's algorithm)



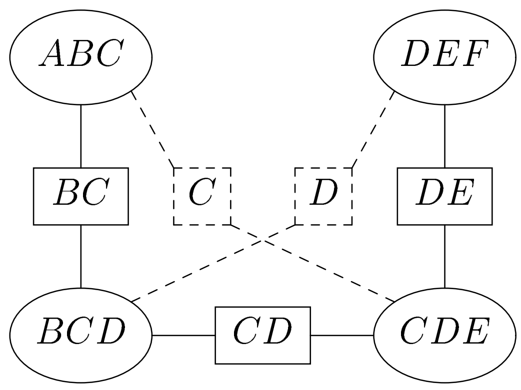
DAG



Moral graph



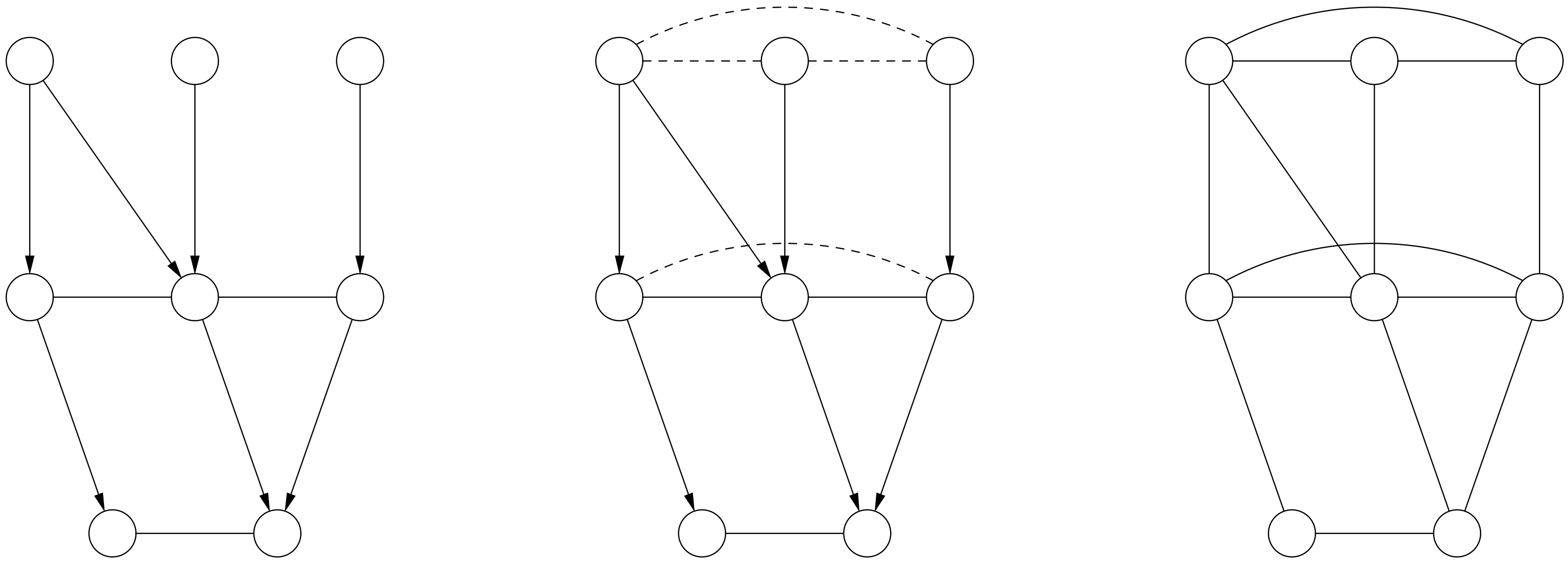
Triangulated graph



Junction tree

Construction of junction trees

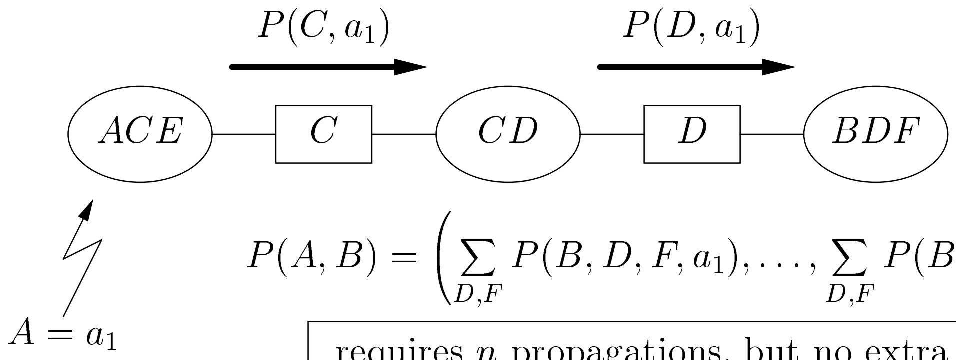
Chain graphs



Computing arbitrary marginals

A simple example: What is $\mathbf{P}(\mathbf{A},\mathbf{B})$?

Method 1 (firing of variables — in a consistent junction tree)

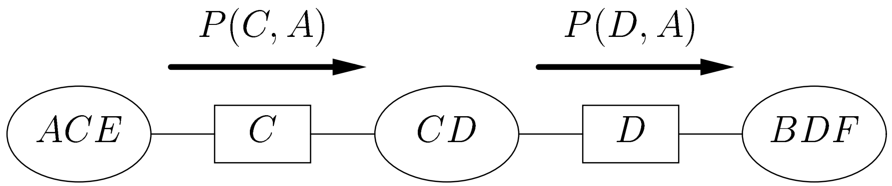


$$P(B,D,F,a_1) = P(B,D,F)\frac{P(D,a_1)}{P(D)}$$

$$P(A,B) = \left(\sum_{D,F} P(B,D,F,a_1),\ldots,\sum_{D,F} P(B,D,F,a_n)\right)$$

$\boxed{\text{requires } n \text{ propagations, but no extra space}}$

Method 2 (propagation of variables — in a consistent junction tree)



$$P(B,D,F,A) = P(B,D,F)\frac{P(D,A)}{P(D)}$$

$$P(D,A) = \sum_{C} P(C,D)\frac{P(C,A)}{P(C)}$$

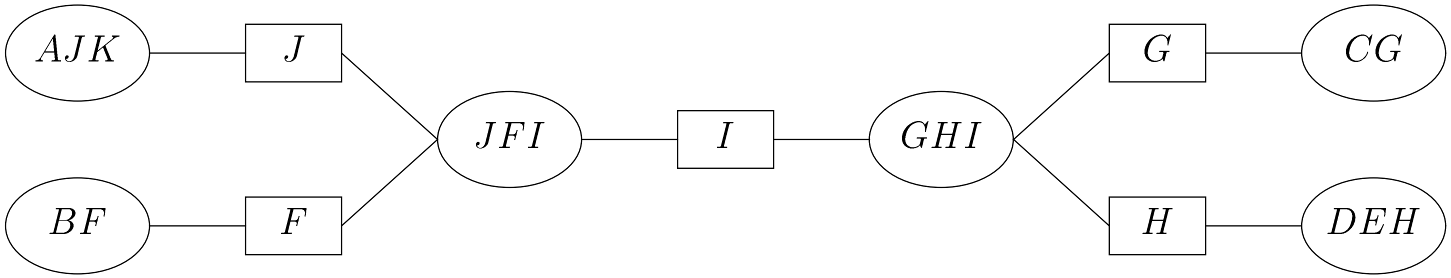
$$P(C,A) = \sum_{E} P(A,C,E)$$

$$P(A,B) = \sum_{D,F} P(B,D,F,A)$$

$\boxed{\text{requires 1 propagation, but some extra space}}$

Computing arbitrary marginals

A slightly more complex example: What is $\mathbf{P(A, B, C, D, E)}$?



Firing of variables: $P(A,B,C,D,E) = \left(\sum_H P(D,E,H,a_1,b_1,c_1), \ldots, \sum_H P(D,E,H,a_n,b_m,c_r) \right)$

$$\boxed{\text{requires } n \times m \times r \text{ propagations, but no extra space}}$$

Propagation of variables: $P(A,B,C,D,E) = \sum_H P(D,E,H) \dfrac{P(H,A,B,C)}{P(H)}$, where

$P(H,A,B,C) = \sum_I \dfrac{P(I,A,B)}{P(I)} \sum_G P(G,H,I) \dfrac{P(G,C)}{P(G)}, \quad P(I,A,B) = \sum_{J,F} P(J,F,I) \dfrac{P(A,J)}{P(J)} \dfrac{P(B,F)}{P(F)}$, etc.

$$\boxed{\text{requires 1 propagation, but 4.4 times more space (binary variables)}}$$

Data conflict

What is it?

Evidence $\mathbf{e}_1$ and evidence $\mathbf{e}_2$ may be

- positively correlated: $P(\mathbf{e}_1 | \mathbf{e}_2) > P(\mathbf{e}_1) \iff P(\mathbf{e}_1, \mathbf{e}_2) > P(\mathbf{e}_1)P(\mathbf{e}_2) \iff \dfrac{P(\mathbf{e}_1)P(\mathbf{e}_2)}{P(\mathbf{e}_1, \mathbf{e}_2)} < 1$
- negatively correlated: $P(\mathbf{e}_1 | \mathbf{e}_2) < P(\mathbf{e}_1) \iff P(\mathbf{e}_1, \mathbf{e}_2) < P(\mathbf{e}_1)P(\mathbf{e}_2) \iff \dfrac{P(\mathbf{e}_1)P(\mathbf{e}_2)}{P(\mathbf{e}_1, \mathbf{e}_2)} > 1$
- independent: $P(\mathbf{e}_1 | \mathbf{e}_2) = P(\mathbf{e}_1) \iff P(\mathbf{e}_1, \mathbf{e}_2) = P(\mathbf{e}_1)P(\mathbf{e}_2) \iff \dfrac{P(\mathbf{e}_1)P(\mathbf{e}_2)}{P(\mathbf{e}_1, \mathbf{e}_2)} = 1$

There is a *conflict* between $\mathbf{e}_1$ and $\mathbf{e}_2$ if

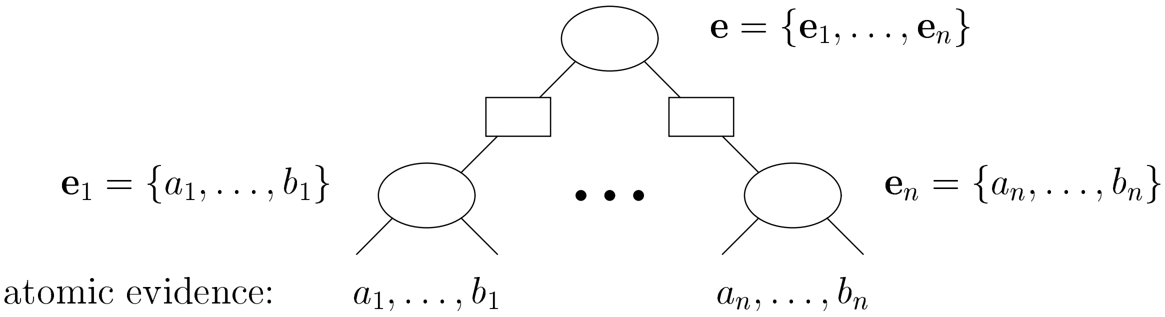
$$\frac{P(\mathbf{e}_1)P(\mathbf{e}_2)}{P(\mathbf{e}_1, \mathbf{e}_2)} > 1 \iff \log \frac{P(\mathbf{e}_1)P(\mathbf{e}_2)}{P(\mathbf{e}_1, \mathbf{e}_2)} > 0,$$

i.e., $\mathbf{e}_1$ and $\mathbf{e}_2$ are negatively correlated. Thus, if $\mathbf{e} = \{\mathbf{e}_1, \mathbf{e}_2\}$,

$$\boxed{\mathrm{conf}(\mathbf{e}) = \log \frac{P(\mathbf{e}_1)P(\mathbf{e}_2)}{P(\mathbf{e}_1, \mathbf{e}_2)}}$$

Data conflict

General definition

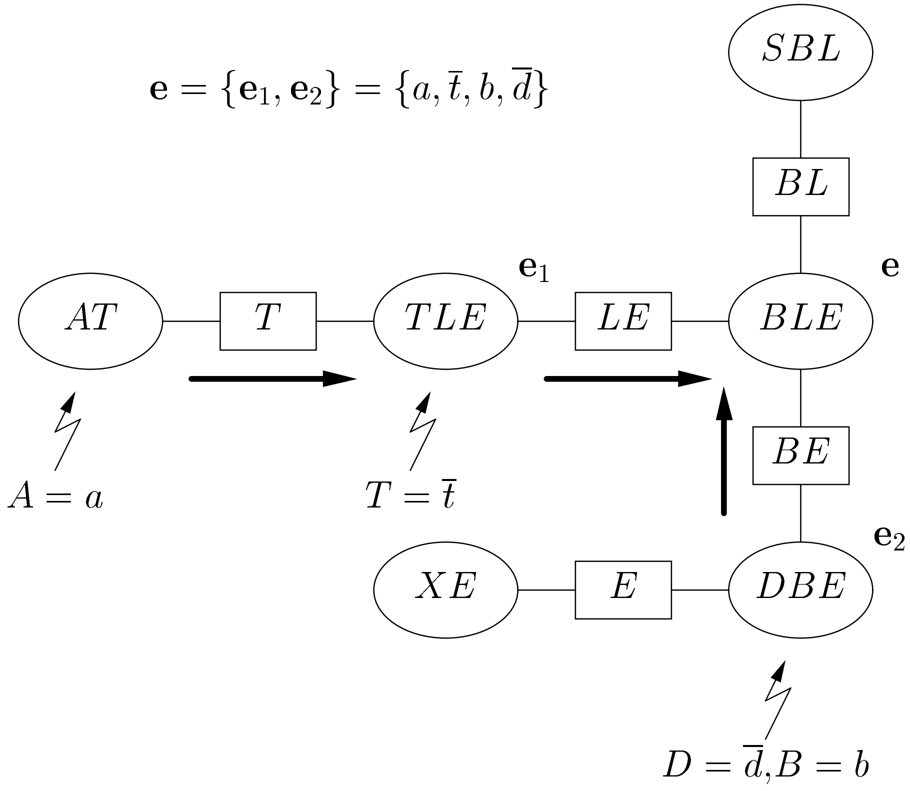


$$
\begin{aligned}
\mathrm{conf}(\mathbf{e}) &= \log \frac{P(a_1)\cdots P(a_n)P(b_1)\cdots P(b_n)}{P(a_1, \ldots, a_n, b_1, \ldots, b_n)} \\
&= \log \frac{P(a_1, \ldots, b_1)\cdots P(a_n, \ldots, b_n)}{P(a_1, \ldots, a_n, b_1, \ldots, b_n)} + \log \frac{P(a_1)\cdots P(b_1)}{P(a_1, \ldots, b_1)} + \cdots + \log \frac{P(a_n)\cdots P(b_n)}{P(a_n, \ldots, b_n)} \\
&= \mathrm{conf}(\mathbf{e}_1, \ldots, \mathbf{e}_n) + \sum_{i=1}^{n} \mathrm{conf}(\mathbf{e}_i)
\end{aligned}
$$

That is, the conflict measure of a clique may be expressed as the sum of a *local* conflict and the corresponding *partial* conflicts.

Data conflict

Tracing a conflict



$p(a) = 0.01$, $p(\bar{t}) = 0.99$, $p(b) = 0.45$, $p(\bar{d}) = 0.564$

$p(\mathbf{e}_1) = 0.0864$, $p(\mathbf{e}_2) = 0.0095$, $p(\mathbf{e}) = 0.0008$

$$\begin{aligned}
\mathrm{conf}(\mathbf{e}) &= \log \frac{p(a)p(\bar{t})p(b)p(\bar{d})}{p(\mathbf{e})} = 1.61 \\
&= \mathrm{conf}(\mathbf{e}_1, \mathbf{e}_2) + \mathrm{conf}(\mathbf{e}_1) + \mathrm{conf}(\mathbf{e}_2)
\end{aligned}$$

$$\begin{aligned}
\mathrm{conf}(\mathbf{e}_1, \mathbf{e}_2) &= \log \frac{p(\mathbf{e}_1)p(\mathbf{e}_2)}{p(\mathbf{e}_1, \mathbf{e}_2)} = -0.007 \\
\mathrm{conf}(\mathbf{e}_1) &= \log \frac{p(a)p(\bar{t})}{p(a, \bar{t})} = -3.13 \\
\mathrm{conf}(\mathbf{e}_2) &= \log \frac{p(b)p(\bar{d})}{p(b, \bar{d})} = 4.74
\end{aligned}$$

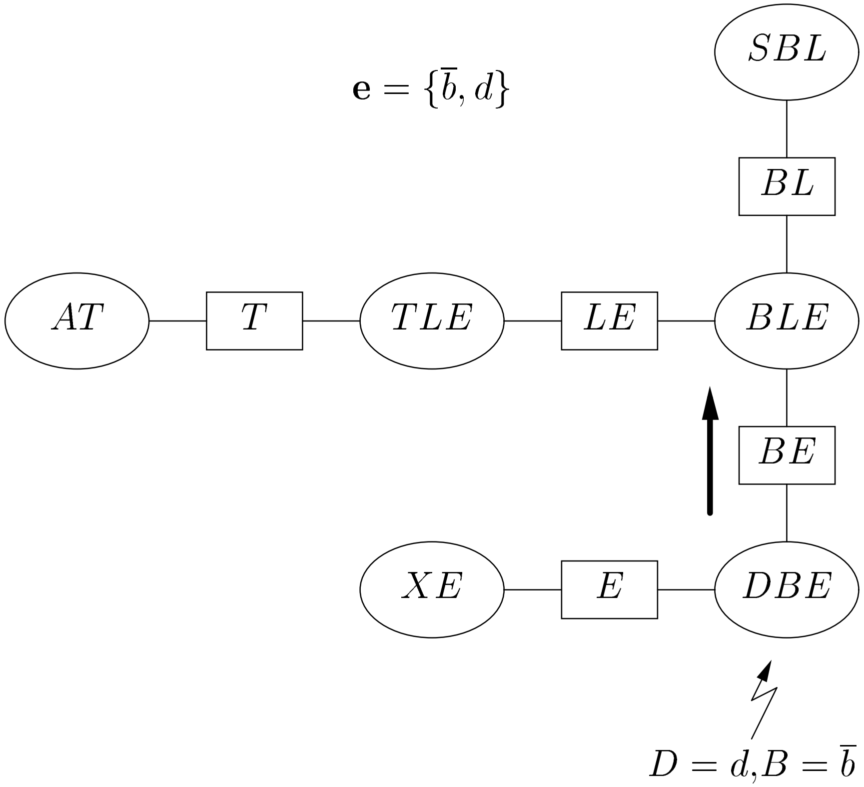
That is, the global conflict is due to local conflict between $B = b$ and $D = \bar{d}$.

NB: $p(\mathbf{e}) = \sum_{B,L,E} p(B, L, E, \mathbf{e})$, $\quad p(\mathbf{e}_1) = \sum_{T,L,E} p(T, L, E, \mathbf{e}_1)$, etc.

Data conflict

Conflict or rare case?

$\mathbf{e} = \{\bar{b}, d\}$



$p(\bar{b}) = 0.55,\ p(d) = 0.436$

$p(\mathbf{e}) = p(\bar{b}, d) = 0.0724$

$$\mathrm{conf}(\mathbf{e}) = \log \frac{p(\bar{b})p(d)}{p(\bar{b}, d)} = 1.73$$

Hypothesis $T = t$ explains $\mathbf{e}$ if

$$\mathrm{conf}(\bar{b}, d, t) = \log \frac{p(\bar{b})p(d)p(t)}{p(\mathbf{e}, t)} \leq 0$$

$$\Updownarrow$$

$$\mathrm{conf}(\mathbf{e}) \leq \log \frac{p(t \,|\, \mathbf{e})}{p(t)} = 2.41$$

That is, the conflict is explained away by $T = t$.

Sensitivity analysis

evidence $\mathbf{e}$

what is the impact of $\mathbf{e}' \subseteq \mathbf{e}$ on h_i?

is $\mathbf{e}' \subseteq \mathbf{e}$ sufficient for h_i?

is $\mathbf{e}' \subseteq \mathbf{e}$ irrelevant for h_i?

does $\mathbf{e}' \subseteq \mathbf{e}$ discriminate between h_i and h_j?

hypotheses $h_1, h_2, \ldots$

Impact of $\mathbf{e}' \subseteq \mathbf{e}$ on h: $\dfrac{p(h\,|\,\mathbf{e}')}{p(h)} = \dfrac{p(\mathbf{e}'\,|\,h)}{p(\mathbf{e}')}$ (normalized likelihoods)

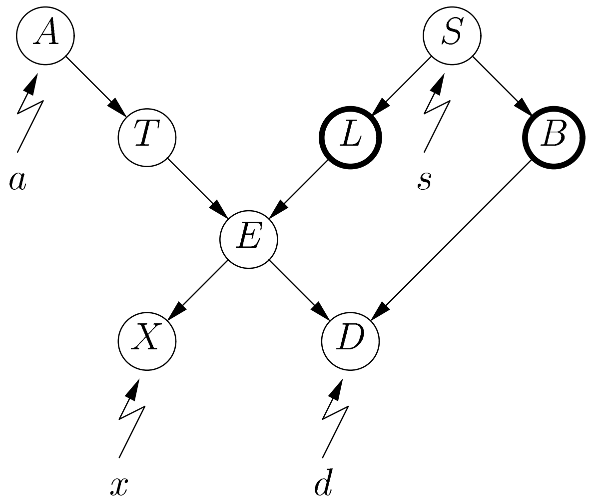
Sufficiency of $\mathbf{e}' \subseteq \mathbf{e}$ for h: $\left|\dfrac{p(h\,|\,\mathbf{e}')}{p(h\,|\,\mathbf{e})} - 1\right| < \delta$

Redundancy of $\mathbf{e}' \subseteq \mathbf{e}$ for h: $\left|\dfrac{p(h\,|\,\mathbf{e} \setminus \mathbf{e}')}{p(h\,|\,\mathbf{e})} - 1\right| < \delta$

Discriminating power of $\mathbf{e}' \subseteq \mathbf{e}$: $\dfrac{p(\mathbf{e}'\,|\,h_i)}{p(\mathbf{e}'\,|\,h_j)}$ (ratio of impacts)

Sensitivity analysis

Example



$$\mathbf{e} = \{a, d, s, x\}$$

How big is the impact of various subsets $\mathbf{e}' \subseteq \mathbf{e}$ on $L = l$ and $B = b$?

How good is the discriminating power of various subsets $\mathbf{e}' \subseteq \mathbf{e}$?

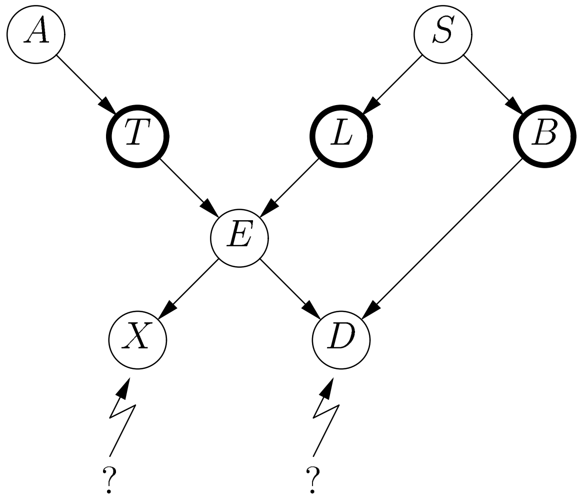
Which subsets $\mathbf{e}' \subseteq \mathbf{e}$ are sufficient for $L = l$ and $B = b$?

Which subsets $\mathbf{e}' \subseteq \mathbf{e}$ are redundant for $L = l$ and $B = b$?

Sensitivity analysis

	impact		discr.	sufficiency		redundancy	
$\mathbf{e}'$	$\frac{p(\mathbf{e}' \mid l)}{p(\mathbf{e}')}$	$\frac{p(\mathbf{e}' \mid b)}{p(\mathbf{e}')}$	$\frac{p(\mathbf{e}' \mid l)}{p(\mathbf{e}' \mid b)}$	$\left\lvert\frac{p(l \mid \mathbf{e}')}{p(l \mid \mathbf{e})} - 1\right\rvert$	$\left\lvert\frac{p(b \mid \mathbf{e}')}{p(b \mid \mathbf{e})} - 1\right\rvert$	$\left\lvert\frac{p(l \mid \mathbf{e} \setminus \mathbf{e}')}{p(l \mid \mathbf{e})} - 1\right\rvert$	$\left\lvert\frac{p(b \mid \mathbf{e} \setminus \mathbf{e}')}{p(b \mid \mathbf{e})} - 1\right\rvert$
$\{a, d, s, x\}$	10.53	1.56	6.76	0.00	0.00	0.91	0.36
$\{a, d, s\}$	2.65	1.93	1.37	0.75	0.24	0.16	0.28
$\{a, d, x\}$	8.08	1.40	5.78	0.23	0.10	0.83	0.14
$\{a, s, x\}$	9.64	1.33	7.23	0.08	0.14	0.82	0.19
$\{d, s, x\}$	13.16	1.59	8.30	0.25	0.02	0.91	0.36
$\{a, d\}$	1.81	1.80	1.00	0.83	0.16	0.12	0.14
$\{a, s\}$	1.82	1.33	1.36	0.83	0.14	0.07	0.03
$\{a, x\}$	6.75	1.09	6.19	0.36	0.30	0.74	0.26
$\{d, s\}$	2.70	1.96	1.38	0.74	0.26	0.36	0.30
$\{d, x\}$	11.30	1.52	7.45	0.07	0.03	0.83	0.14
$\{s, x\}$	11.75	1.33	8.81	0.12	0.14	0.83	0.16
$\{a\}$	1.00	1.00	1.00	0.91	0.36	0.25	0.02
$\{d\}$	1.87	1.85	1.01	0.82	0.19	0.08	0.14
$\{s\}$	1.82	1.33	1.36	0.83	0.14	0.23	0.10
$\{x\}$	8.89	1.13	7.90	0.16	0.28	0.75	0.24
$\{\}$	1.00	1.00	1.00	0.91	0.36	0.00	0.00

Value of information



What is the expected profit for hypothesis H of performing test T?

Expected value of T:

$$EV(T) = \sum_{t \in T} V(p(H \mid t))p(t)$$

Expected benefit of T:

$$EB(T) = EV(T) - V(p(H))$$

Expected profit of T:

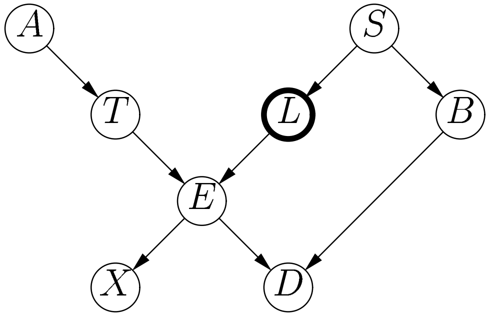
$$EP(T) = EB(T) - C_T$$

Value of information

Utility-based value function

Value function = expected utility of optimal action: $V(p(H)) = \max_{a \in A} \sum_{h \in H} U(a,h)p(h)$

Example



Actions

L	no x-ray	x-ray
l	-100	100
$\bar{l}$	-10	-50

$H = L$, $T = X$, $C_T = 5$

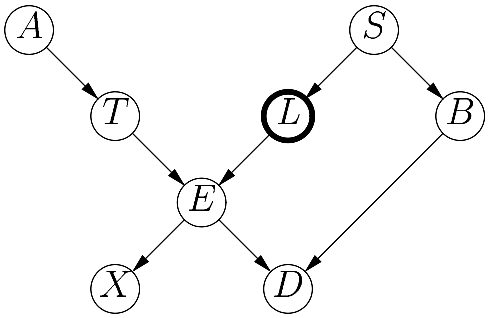
$$\begin{aligned}
V(p(L)) &= \max\{U(\text{no x-ray}, l)p(l) + U(\text{no x-ray}, \bar{l})p(\bar{l}),\\
&\qquad U(\text{x-ray}, l)p(l) + U(\text{x-ray}, \bar{l})p(\bar{l})\}\\
&= \max\{-100 \cdot 0.055 + 10 \cdot 0.945, 100 \cdot 0.055 - 50 \cdot 0.945\} = 4\\
V(p(L \,|\, x)) &= \max\{U(\text{no x-ray}, l)p(l \,|\, x) + U(\text{no x-ray}, \bar{l})p(\bar{l} \,|\, x),\\
&\qquad U(\text{x-ray}, l)p(l \,|\, x) + U(\text{x-ray}, \bar{l})p(\bar{l} \,|\, x)\}\\
&= \max\{-100 \cdot 0.489 + 10 \cdot 0.511, 100 \cdot 0.489 - 50 \cdot 0.511\} = 23.4\\
V(p(L \,|\, \bar{x})) &= \max\{U(\text{no x-ray}, l)p(l \,|\, \bar{x}) + U(\text{no x-ray}, \bar{l})p(\bar{l} \,|\, \bar{x}),\\
&\qquad U(\text{x-ray}, l)p(l \,|\, \bar{x}) + U(\text{x-ray}, \bar{l})p(\bar{l} \,|\, \bar{x})\}\\
&= \max\{-100 \cdot 0.001 + 10 \cdot 0.999, 100 \cdot 0.001 - 50 \cdot 0.999\} = 9.9\\
EV(X) &= V(p(L \,|\, x))p(x) + V(p(L \,|\, \bar{x}))p(\bar{x})\\
&= 23.4 \cdot 0.1103 + 9.9 \cdot 0.8897 = 11.4\\
EB(X) &= EV(X) - V(p(L)) = 11.4 - 4 = 7.4\\
EP(X) &= EB(X) - C_X = 7.4 - 5 = 2.4
\end{aligned}$$

Value of information

Entropy-based value function

Value function = negative entropy of $p(H)$: $V(p(H)) = \sum_{h \in H} p(h) \log_2 p(h)$

Example



$H = L$, $T = X$, $C_T = 0.15$

$$
\begin{aligned}
V(p(L)) &= p(l)\log_2 p(l) + p(\bar{l})\log_2 p(\bar{l}) \\
&= 0.055 \cdot \log_2 0.055 + 0.945 \cdot \log_2 0.945 = -0.3073 \\
V(p(L \,|\, x)) &= p(l \,|\, x)\log_2 p(l \,|\, x) + p(\bar{l} \,|\, x)\log_2 p(\bar{l} \,|\, x) \\
&= 0.489 \cdot \log_2 0.489 + 0.511 \cdot \log_2 0.511 = -1 \\
V(p(L \,|\, \overline{x})) &= p(l \,|\, \overline{x})\log_2 p(l \,|\, \overline{x}) + p(\bar{l} \,|\, \overline{x})\log_2 p(\bar{l} \,|\, \overline{x}) \\
&= 0.001 \cdot \log_2 0.001 + 0.999 \cdot \log_2 0.999 = -0.0138 \\
EV(X) &= V(p(L \,|\, x))p(x) + V(p(L \,|\, \overline{x}))p(\overline{x}) \\
&= -1 \cdot 0.1103 - 0.0138 \cdot 0.8897 = -0.1225 \\
EB(X) &= EV(X) - V(p(L)) = -0.1225 - (-0.3073) = 0.1848 \\
EP(X) &= EB(X) - C_X = 0.1848 - 0.15 = 0.03
\end{aligned}
$$