A Quantitative Characterization of Weighted Kripke Structures in Temporal Logic.

Uli Fahrenberg and Kim G. Larsen and Claus Thrane

Dept. Computer Science

AALBORG UNIVERSITY

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Related work

- Brown, Clarke, Grömberg: Characterising Kripke Structures in Temporal Logic.
- Chatterjee, Doyern and Henzinger: Quantitative Languages.
- Henzinger, Majumdar, and Prabhu: Quantifying Similarities Between Timed Systems.
- Alfaro, Faella, and Stoelinga: Linear and Branching Metrics for Quantitative Transition Systems.

Strong ties to probabilistic systems and weighted automata.

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The quantitative model-checking problem

Given a state s of a structire M, and a logical formulae φ

- Does φ hold at s? **i.e.** $M, s \models \varphi$?

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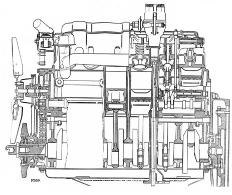
Given a state s of a structire M, and a logical formulae φ

- What is the degree ε with which φ holds at s?
  i.e. compute ε s.t. $[\![\varphi]\!]_M(s) = \varepsilon$?

Why deal with quantitative model-checking?



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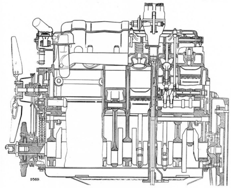

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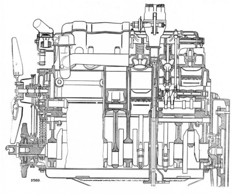

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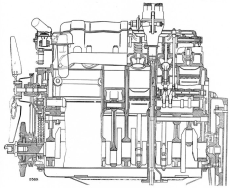

Decidability & Expressiveness

Model-checking

Given φ and $s \in S$: $s \models \varphi$?

Satisfyability

Given φ: $\exists s \in S . s \models \varphi$?

Adequacy

Given $s, t \in S$: $s \sim t$ iff $\forall \varphi . s \models \varphi \iff t \models \varphi$?

Characteristic properties

Given $s, t \in S$: $t \models \varphi_s \iff s \sim t$?

Decidability & Expressiveness

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Given φ and $s \in S$: $[\![\varphi]\!](s) = \varepsilon$?

Satisfyability

Given φ: $\exists s \in S.[\![\varphi]\!](s) = 0$?

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Given $s, t \in S$: $s \sim_\varepsilon t$ iff $\forall \varphi.|[\![\varphi]\!](s) - [\![\varphi]\!](t)| \leq \varepsilon$?

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Given $s, t \in S$: $[\![\varphi_s]\!](t) = \varepsilon \iff t \sim_\varepsilon s$?

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Given $s, t \in S$: $[\![\varphi_s]\!](t) = \varepsilon \iff t \sim_\varepsilon s$?

Weighted Kripke structure

For a finite set $\mathcal{AP}$ of atomic propositions, a **weighted Kripke structure** is a quadruple $M = (S, T, \mathcal{L}, w)$ where

- S is a finite set of states,
- $T \subseteq S \times S$ is a transition relation
- $\mathcal{L} : S \to 2^{\mathcal{AP}}$ is the proposition labelling, and
- $w : T \to \mathbb{R}_{\geq 0}$ assigns weights to transitions.

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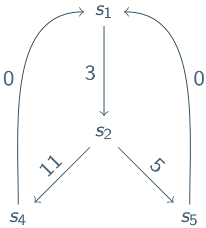

Bisimulation for WKS

Let $(S, T, \mathcal{L}, w)$ be a WKS on a set $\mathcal{AP}$ of atomic propositions. A relation $B \subseteq S \times S$ is a weighted bisimulation relation, provided that for all $(s, t) \in B$:

- $\mathcal{L}(s) = \mathcal{L}(t)$ and
- if $s \xrightarrow{c} s'$, then also $t \xrightarrow{c} t'$ where $(s', t') \in B$ for some $t' \in S'$,
- if $t \xrightarrow{c} t'$, also also $s \xrightarrow{c} s'$ where $(s', t') \in B$ for some $s' \in S$;

We say say that s and t are weighted bisimular, written $s \sim t$ if $(s, t) \in B$ for some unweighted bisimulation B.

Bisimulation for WKS

Let $(S, T, \mathcal{L}, w)$ be a WKS on a set $\mathcal{AP}$ of atomic propositions. A relation $B \subseteq S \times S$ is a unweighted bisimulation relation, provided that for all $(s, t) \in B$:

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- if $s \longrightarrow s'$, then also $t \longrightarrow t'$ where $(s', t') \in B$ for some $t' \in S'$,
- if $t \longrightarrow t'$, also also $s \longrightarrow s'$ where $(s', t') \in B$ for some $s' \in S$;

We say say that s and t are unweighted bisimular, written $s \overset{u}{\sim} t$ if $(s, t) \in B$ for some unweighted bisimulation B.

Bridging the Boolean gap

By definition membership og a relation, is **true** or **false**; We provide a family of relations st.

$$\sim \;\supseteq \cdots \supseteq\; \sim_i \;\supseteq\; \sim_j \;\supseteq \cdots \supseteq\; \overset{u}{\sim}$$

For $i, j \in \mathbb{R}_{\geq 0}$ and $i < j$

Equivalence relations

Definition

Given a set X a binary relation $R \subseteq X \times X$ is an equivalence relation, if and only if, $\forall x, y, z \in X$, R is

- $x\ R\ x$
- $x\ R\ y \iff y\ R\ x$
- $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$

...or

Given a set X a map $R : X \times X \to \{0, 1\}$ is an equivalence relation, if and only if, $\forall x, y, z \in X$, R and:

- $R(x, x) = 0$
- $R(x, y) = R(y, x)$
- $R(x, y) \leq R(y, z) + R(y, z)$

Where $\mathbf{tt} = 0$ and $\mathbf{ff} = 1$

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Boolean

Metrics

Definition

Given a set X. Then a metric on X is a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ which $\forall x, y, z \in X$ satisfies:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Distances on sequences of real numbers

For sequences $a = (a_i)$, $b = (b_i)$, we may consider the following distances:

$$d_{\bullet}(a, b) = \sup_i \left\{ |a_i - b_i| \right\}$$

$$d_{+}(a, b) = \sum_i |a_i - b_i|$$

$$d_{\pm}(a, b) = \sup_i \left\{ \left| \sum_{j=0}^{i} a_j - \sum_{j=0}^{i} b_j \right| \right\} \qquad \big([\text{HMP'05}]\big)$$

Quantifying bisimulation

We extend bisimulation, with $d_\bullet$, d_+ or $d_\pm$ measurements, as well as a **discounting** factor $0 \leq \lambda \leq 1$.

- Point-wise (bi)simulation
- Accumulated (bi)simulation
- Max-lead (bi)simulation (Henzinger et. al, FORMATS'05 to be poly-time[a] decidable for timed automata)

[a] in the size of the region graph, which in turn is exponential in the size of clocks

A **bisimulation distance** is a function $d : S \times S \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ which satisfies the following for all $s_1, s_2, s_3 \in S$:

- $d(s_1, s_1) = 0$,
- $d(s_1, s_2) + d(s_2, s_3) \geq d(s_1, s_3)$,
- $d(s_1, s_2) = d(s_2, s_1)$,
- $s_1 \sim s_2$ implies $d(s_1, s_2) = 0$
- $d(s_1, s_2) \neq \infty$ implies $s_1 \overset{u}{\sim} s_2$

Accumulated bisimulation distance

A family of relations $\mathbf{R} = \{\mathcal{R}_\varepsilon \subseteq S \times S \mid \varepsilon > 0\}$

Accumulated bisimulation distance

A family of relations $\mathbf{R} = \{\mathcal{R}_\varepsilon \subseteq S \times S \mid \varepsilon > 0\}$ on a WKS $(S, T, \mathcal{L}, w)$ is an **accumulating bisimulation family** provided that for all $(s, t) \in \mathcal{R}_\varepsilon \in \mathbf{R}$:

- $\mathcal{L}(s) = \mathcal{L}(t)$ and
- for all $s \xrightarrow{c} s'$, also $t \xrightarrow{d} t'$ with $|c - d| \leq \varepsilon$ for some $d \in \mathbb{R}_{\geq 0}$ and $(s', t') \in \mathcal{R}_{\varepsilon'} \in \mathbf{R}$ with $\varepsilon' \leq \frac{\varepsilon - |c-d|}{\lambda}$,
- for all $t \xrightarrow{c} t'$, also $s \xrightarrow{d} s'$ with $|c - d| \leq \varepsilon$ for some $d \in \mathbb{R}_{\geq 0}$ and $(s', t') \in \mathcal{R}_{\varepsilon'} \in \mathbf{R}$ with $\varepsilon' \leq \frac{\varepsilon - |c-d|}{\lambda}$.

We write $s \overset{+}{\sim}_\varepsilon t$ if $(s, t) \in \mathcal{R}_\varepsilon \in \mathbf{R}$ for an accumulating bisimulation family $\mathbf{R}$.

Linear and Branching d_+-distances

(simulation as equations)

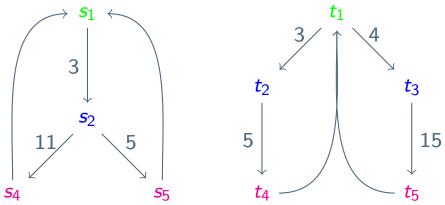
For states $s, t \in S$, the **accumulated branching distances** is the minimal fixed points to the following recursive equation:

$$\wr s, t \wr_+ = \sup_{s \xrightarrow{c} s'} \inf_{t \xrightarrow{d} t'} |c - d| + \lambda \cdot \wr s', t' \wr_+$$

and the **accumulated linear distance** is:

$$|s, t|_+ = \sup_{\sigma \in \mathrm{P}(s)} \inf_{\sigma' \in \mathrm{P}(t)} \sum_i \lambda^i |\sigma(i)_w - \sigma'(i)_w|$$

Example



For $\lambda = 0.9$ $\mathcal{AP} = \{\bullet, \bullet, \bullet\}$ Simulation distances:

$$|s_1, t_1|_+ = \sum_i (1 + 4\lambda)\lambda^{3i} \approx 17.0$$

$$\wr s_1, t_1 \wr_+ = 1 + 10\lambda + \lambda^3 \wr s_1, t_1 \wr_+ \approx 36.9$$

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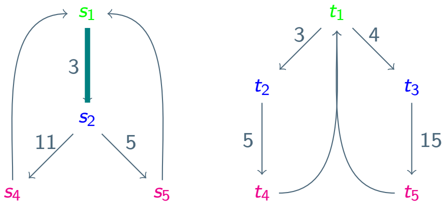

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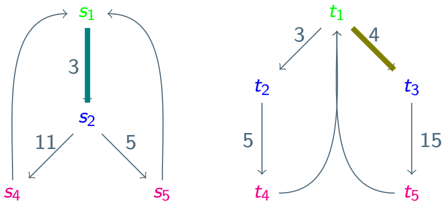

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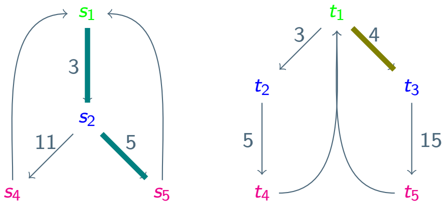

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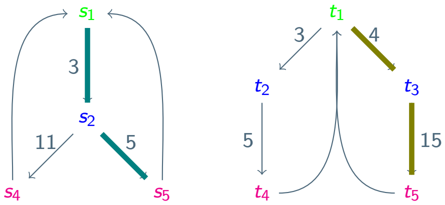

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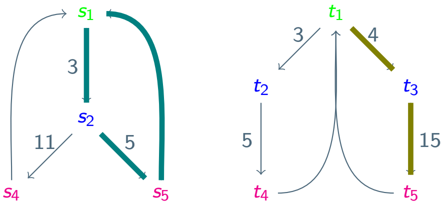

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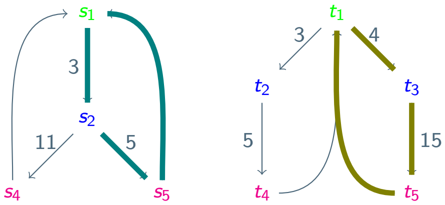

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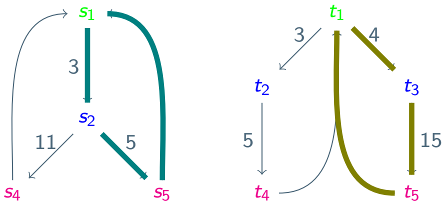

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And w.r.t their accumulating bisimulation distance; $s_1 \overset{+}{\sim}_{37} t_1$.

Properties

- For all states $s, t \in S$, we have

$$|s,t|_{\bullet} \leq \wr s,t \wr_{\bullet} \qquad |s,t|_{+} \leq \wr s,t \wr_{+} \qquad |s,t|_{\pm} \leq \wr s,t \wr_{\pm}$$

- The distances $|\cdot,\cdot|_{\bullet}$ and $\wr \cdot,\cdot \wr_{\bullet}$ are topologically inequivalent. Similarly, $|\cdot,\cdot|_{+}$ and $\wr \cdot,\cdot \wr_{+}$, and also $|\cdot,\cdot|_{\pm}$ and $\wr \cdot,\cdot \wr_{\pm}$, are topologically inequivalent.

Computability for Weighted Timed Automata

Theorem

For discounting factor $\lambda < 1$ and $|\cdot, \cdot|$ any of the three **trace distances**, *it is undecidable whether $|s, t| = 0$ for weighted timed automata.*

Theorem

For discounting factor $\lambda < 1$, **accumulating branching distance** *from deterministic to non-deterministic* **weighted timed automata** *is computable.*

Setting the scene for the logic

- A simple general syntactic extension of CTL,
- The semantics of a formulae φ defines a map $[\![\cdot]\!] : S \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$. such that
  - The state semantics are shared for $d_{\cdot}$ and d_{+} and $d_{\pm}$.
  - The path semantics are specific the respective distances.
- To obtain a correspondence, with the **bisimulation distance**.

Weighted CTL

For any of the metrics $d_{\cdot}$ and d_+ and $d_\pm$, we define the syntax:

Definition

For $p \in \mathcal{AP}$, Φ generates the set of state formulae, and Ψ, the set of path formulae, annotated by weights $c \in \mathbb{R}_{\geq 0}$, according to the following abstract syntax:

$$\Phi ::= p \mid \neg p \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid E\Psi \mid A\Psi$$
$$\Psi ::= X_c\Phi \mid G_c\Phi \mid F_c\Phi \mid [\Phi_1 U_c \Phi_2]$$

The logic WCTL is the set of state formulae, which we denote $\mathcal{L}_w(\mathcal{AP})$ or simply $\mathcal{L}_w$.

Semantics for state formulae

Let $\varphi, \varphi_1, \varphi_2$ be state formulae and ψ a path formula. The valuation $[\![\cdot]\!] : S \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is defined inductively.

$$[\![p]\!](s) = \begin{cases} 0 & \text{if } p \in \mathcal{L}(s) \\ \infty & \text{otherwise} \end{cases}$$

$$[\![\neg p]\!](s) = \begin{cases} 0 & \text{if } p \in \mathcal{AP} \setminus \mathcal{L}(s) \\ \infty & \text{otherwise} \end{cases}$$

$$[\![\varphi_1 \vee \varphi_2]\!](s) = \inf\left\{[\![\varphi_1]\!](s), [\![\varphi_2]\!](s)\right\}$$
$$[\![\varphi_1 \wedge \varphi_2]\!](s) = \sup\left\{[\![\varphi_1]\!](s), [\![\varphi_2]\!](s)\right\}$$

$$[\![E\psi]\!](s) = \inf\left\{[\![\psi]\!](\sigma) \mid \sigma \in P(s)\right\}$$
$$[\![A\psi]\!](s) = \sup\left\{[\![\psi]\!](\sigma) \mid \sigma \in P(s)\right\}$$

d_+ Path semantics

$$[\![\varphi]\!](\sigma) = [\![\varphi]\!](\sigma(0)_s)$$

$$[\![X_c\varphi]\!](\sigma) = |c - \sigma(0)_w| + \lambda[\![\varphi]\!](\sigma^1)$$

$$[\![F_c\varphi]\!](\sigma) = \inf_k \left(\left| \sum_{j=0}^{k-1} \lambda^j \sigma(j)_w - c \right| + \lambda^k [\![\varphi]\!](\sigma^k) \right)$$

$$[\![G_c\varphi]\!](\sigma) = \sup_k \left(\left| \sum_{j=0}^{k-1} \lambda^j \sigma(j)_w - c \right| + \lambda^k [\![\varphi]\!](\sigma^k) \right)$$

$$[\![\varphi_1 U_c \varphi_2]\!](\sigma) = \inf_k \left(\left| \sum_{j=0}^{k-1} \lambda^j [\![\varphi_1]\!](\sigma^j) - c \right| + \lambda^k [\![\varphi_2]\!](\sigma^k) \right)$$

Properties of the logic

Theorem: Adequacy

For states $s, t \in S$, $s \overset{+}{\sim}_\varepsilon t$ if and only if $\big|[\![\varphi]\!](s) - [\![\varphi]\!](t)\big| \leq \varepsilon$ for all $\varphi \in \mathcal{L}_w$.

Theorem: Expressivity & Characteristic formulae

For each $s \in S$ and every $\gamma \in \mathbb{R}_+$, there exists a state formula $\varphi^s_\gamma \in \mathcal{L}_w$ which characterizes s up to accumulating bisimulation and up to γ, **i.e.** such that for all $s' \in S$, $s \overset{+}{\sim}_\varepsilon s'$ if and only if $[\![\varphi^s_\gamma]\!](s') \in [\varepsilon - \gamma, \varepsilon + \gamma]$ for all γ.

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Building Characteristic formulae

HML for infinite state systems [Graf, Sifakis '86]

Use recursive properties

CTL for finite state systems [Brown, Clarke, Grömberg '87]

Use the characteristic number c

WCTL for accumulating quantitative checking

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WCTL for accumulating quantitative checking

For infinite sums, the constant c, is insufficient

d_+ Characteristic formulae

For each $s \in S$ and $n \in \mathbb{N}$, denote $\mathcal{L}(s) = \{\mathsf{p}_1, \ldots, \mathsf{p}_k\}$ and $\mathcal{AP} \setminus \mathcal{L}(s) = \{\mathsf{q}_1, \ldots, \mathsf{q}_\ell\}$ the formula $\varphi(s, n)$ is defined inductively as:

$$\varphi(s,0) = (\mathsf{p}_1 \wedge \cdots \wedge \mathsf{p}_k) \wedge (\neg \mathsf{q}_1 \wedge \cdots \wedge \neg \mathsf{q}_\ell)$$

$$\varphi(s,n+1) = \bigwedge_{s \xrightarrow{w} s'} \mathsf{EX}_w \varphi(s',n) \wedge \bigwedge_{w : s \xrightarrow{w} s'} \mathsf{AX}_w \Big(\bigvee_{s \xrightarrow{w} s'} \varphi(s',n) \Big) \wedge \varphi(s,0)$$

d_+ Characteristic formulae

For each $s \in S$ and $n \in \mathbb{N}$, denote $\mathcal{L}(s) = \{p_1, \ldots, p_k\}$ and $\mathcal{AP} \setminus \mathcal{L}(s) = \{q_1, \ldots, q_\ell\}$ the formula $\varphi(s, n)$ is defined inductively as:

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- It is easy to see that $[\![\varphi(s,n)]\!](s) = 0$ for all n.

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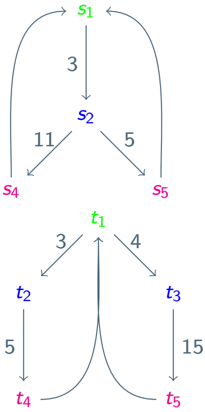
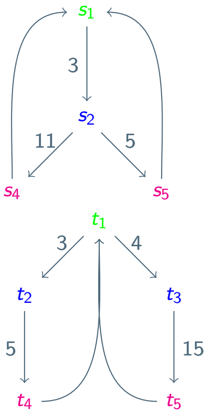
- It is easy to see that $[\![\varphi(s, n)]\!](s) = 0$ for all n.
- Observe that for each $\gamma > 0$, there is $n(\gamma) \in \mathbb{N}$ such that $\varphi(s, n(\gamma))$ can play the role of φ^s_γ in the theorem.

Example



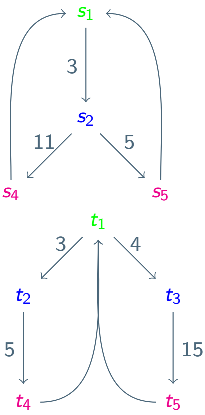
$$\varphi(t_1, n) = (\textcolor{green}{\bullet} \wedge \neg \textcolor{blue}{\bullet} \wedge \neg \textcolor{magenta}{\bullet}) \wedge \mathrm{EX}_3\varphi(t_2, n-1) \wedge \mathrm{EX}_4\varphi(t_3, n-1) \wedge \mathrm{AX}_3\varphi(t_2, n-1) \wedge \mathrm{AX}_4\varphi(t_3, n-1)$$

Example



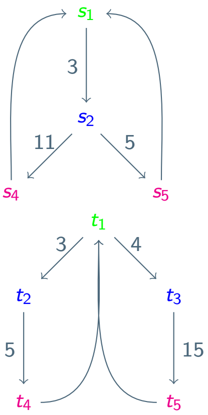
$$
\begin{aligned}
\varphi(t_1, n) = (&{\color{green}\bullet} \wedge \neg{\color{blue}\bullet} \wedge \neg{\color{magenta}\bullet}) \\
&\wedge \mathrm{EX}_3\varphi(t_2, n-1) \wedge \mathrm{EX}_4\varphi(t_3, n-1) \\
&\wedge \mathrm{AX}_3\varphi(t_2, n-1) \wedge \mathrm{AX}_4\varphi(t_3, n-1) \\
\varphi(t_2, n-1) = (&\neg{\color{green}\bullet} \wedge {\color{blue}\bullet} \wedge \neg{\color{magenta}\bullet}) \\
&\wedge \mathrm{EX}_5\varphi(t_4, n-2) \wedge \mathrm{AX}_5\varphi(t_4, n-2)
\end{aligned}
$$

Example



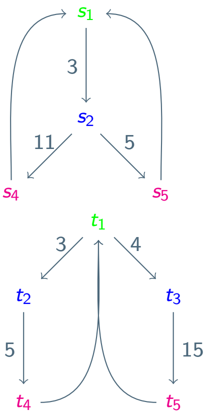
$$\varphi(t_1, n) = (\color{green}\bullet \color{black}\wedge \neg \color{blue}\bullet \color{black}\wedge \neg \color{magenta}\bullet\color{black})$$
$$\wedge\, EX_3\varphi(t_2, n-1) \wedge EX_4\varphi(t_3, n-1)$$
$$\wedge\, AX_3\varphi(t_2, n-1) \wedge AX_4\varphi(t_3, n-1)$$
$$\varphi(t_2, n-1) = (\neg \color{green}\bullet \color{black}\wedge \color{blue}\bullet \color{black}\wedge \neg \color{magenta}\bullet\color{black})$$
$$\wedge\, EX_5\varphi(t_4, n-2) \wedge AX_5\varphi(t_4, n-2)$$
$$\varphi(t_3, n-1) = \ldots$$

Example



$$
\begin{aligned}
\varphi(t_1, n) &= (\bullet \wedge \neg\bullet \wedge \neg\bullet) \\
&\quad \wedge EX_3\varphi(t_2, n-1) \wedge EX_4\varphi(t_3, n-1) \\
&\quad \wedge AX_3\varphi(t_2, n-1) \wedge AX_4\varphi(t_3, n-1) \\
\varphi(t_2, n-1) &= (\neg\bullet \wedge \bullet \wedge \neg\bullet) \\
&\quad \wedge EX_5\varphi(t_4, n-2) \wedge AX_5\varphi(t_4, n-2) \\
\varphi(t_3, n-1) &= \dots \\
\varphi(t_4, n-2) &= \dots
\end{aligned}
$$

Example



$$
\begin{aligned}
\varphi(t_1, n) &= (\textcolor{green}{\bullet} \wedge \neg \textcolor{blue}{\bullet} \wedge \neg \textcolor{magenta}{\bullet}) \\
&\wedge \mathrm{EX}_3 \varphi(t_2, n-1) \wedge \mathrm{EX}_4 \varphi(t_3, n-1) \\
&\wedge \mathrm{AX}_3 \varphi(t_2, n-1) \wedge \mathrm{AX}_4 \varphi(t_3, n-1) \\
\varphi(t_2, n-1) &= (\neg \textcolor{green}{\bullet} \wedge \textcolor{blue}{\bullet} \wedge \neg \textcolor{magenta}{\bullet}) \\
&\wedge \mathrm{EX}_5 \varphi(t_4, n-2) \wedge \mathrm{AX}_5 \varphi(t_4, n-2) \\
\varphi(t_3, n-1) &= \dots \\
\varphi(t_4, n-2) &= \dots \\
\varphi(t_5, n-2) &= \dots
\end{aligned}
$$

Example (cont.)

For $[\![\varphi^{t_1}_{0.1}]\!](s_1)$ we compute:

$$[\![\varphi(t_1, 56)]\!](s_1) = \max \begin{cases} \max \begin{cases} \inf\{|3 - \sigma(0)_w| + \lambda[\![\varphi(t_2, 55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \\ \inf\{|4 - \sigma(0)_w| + \lambda[\![\varphi(t_3, 55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \end{cases} \\ \max \begin{cases} \sup\{|3 - \sigma(0)_w| + \lambda[\![\varphi(t_2, 55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \\ \sup\{|4 - \sigma(0)_w| + \lambda[\![\varphi(t_3, 55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \end{cases} \\ \max\{[\![\bullet]\!](s_1), [\![\neg\bullet]\!](s_1), [\![\neg\bullet]\!](s_1)\} \end{cases}$$

Example (cont.)

For $[\![\varphi^{t_1}_{0.1}]\!](s_1)$ we compute:

$$[\![\varphi(t_1,56)]\!](s_1) = \max \begin{cases} \max \begin{cases} \inf\{|3-\sigma(0)_w| + \lambda[\![\varphi(t_2,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \\ \inf\{|4-\sigma(0)_w| + \lambda[\![\varphi(t_3,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \end{cases} \\ \max \begin{cases} \sup\{|3-\sigma(0)_w| + \lambda[\![\varphi(t_2,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \\ \sup\{|4-\sigma(0)_w| + \lambda[\![\varphi(t_3,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \end{cases} \\ \max\{[\![\bullet]\!](s_1), [\![\neg\bullet]\!](s_1), [\![\neg\bullet]\!](s_1)\} \end{cases}$$

$$[\![\varphi(t_3,55)]\!](\sigma = s_1 \xrightarrow{3} s_2 \xrightarrow{5} s_5 \xrightarrow{0} s_1 \cdots) =$$

Example (cont.)

For $[\![\varphi^{t_1}_{0.1}]\!](s_1)$ we compute:

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$$[\![\varphi(t_3,55)]\!](\sigma = s_1 \xrightarrow{3} s_2 \xrightarrow{5} s_5 \xrightarrow{0} s_1 \cdots) =$$

$$[\![\varphi(t_3,55)]\!](s_2) = \max \begin{cases} \inf\{|15-\sigma(0)_w| + \lambda[\![\varphi(t_5,55)]\!](\sigma^1) \mid \sigma \in \mathsf{P}(s_2)\} \\ \sup\{|15-\sigma(0)_w| + \lambda[\![\varphi(t_5,54)]\!](\sigma^1) \mid \sigma \in \mathsf{P}(s_2)\} \\ \max\{[\![\neg\textcolor{green}{\bullet}]\!](s_2), [\![\textcolor{blue}{\bullet}]\!](s_2), [\![\neg\textcolor{magenta}{\bullet}]\!](s_2)\} \end{cases}$$

Example (cont.)

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$$[\![\varphi(t_3,55)]\!](\sigma = s_1 \xrightarrow{3} s_2 \xrightarrow{5} s_5 \xrightarrow{0} s_1 \cdots) =$$

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$\dots$

Example (cont.)

For $[\![\varphi^{t_1}_{0.1}]\!](s_1)$ we compute:

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$$[\![\varphi(t_3,55)]\!](\sigma = s_1 \xrightarrow{3} s_2 \xrightarrow{5} s_5 \xrightarrow{0} s_1 \cdots) =$$

$$[\![\varphi(t_3,55)]\!](s_2) = \max \begin{cases} \inf\{|15-\sigma(0)_w| + \lambda[\![\varphi(t_5,55)]\!](\sigma^1) \mid \sigma \in \mathsf{P}(s_2)\} \\ \sup\{|15-\sigma(0)_w| + \lambda[\![\varphi(t_5,54)]\!](\sigma^1) \mid \sigma \in \mathsf{P}(s_2)\} \\ \max\{[\![\neg\bullet]\!](s_2), [\![\bullet]\!](s_2), [\![\neg\bullet]\!](s_2)\} \end{cases}$$

$$\ldots$$

$$[\![\varphi(t_5,54)]\!](s_4) = \max \ldots$$

Example (cont.)

For $[\![\varphi^{t_1}_{0.1}]\!](s_1)$ we compute: ≈ 36.809

$$[\![\varphi(t_1,56)]\!](s_1) = \max \begin{cases} \max \begin{cases} \inf\{|3-\sigma(0)_w| + \lambda[\![\varphi(t_2,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \\ \inf\{|4-\sigma(0)_w| + \lambda[\![\varphi(t_3,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \end{cases} \\ \max \begin{cases} \sup\{|3-\sigma(0)_w| + \lambda[\![\varphi(t_2,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \\ \sup\{|4-\sigma(0)_w| + \lambda[\![\varphi(t_3,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_1)\} \end{cases} \\ \max\{[\![\bullet]\!](s_1), [\![\neg\bullet]\!](s_1), [\![\neg\bullet]\!](s_1)\} \end{cases}$$

$$[\![\varphi(t_3,55)]\!](\sigma = s_1 \xrightarrow{3} s_2 \xrightarrow{5} s_5 \xrightarrow{0} s_1 \cdots) =$$

$$[\![\varphi(t_3,55)]\!](s_2) = \max \begin{cases} \inf\{|15-\sigma(0)_w| + \lambda[\![\varphi(t_5,55)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_2)\} \\ \sup\{|15-\sigma(0)_w| + \lambda[\![\varphi(t_5,54)]\!](\sigma^1) \mid \sigma \in \mathrm{P}(s_2)\} \\ \max\{[\![\neg\bullet]\!](s_2), [\![\bullet]\!](s_2), [\![\neg\bullet]\!](s_2)\} \end{cases}$$

$$\ldots$$

$$[\![\varphi(t_5,54)]\!](s_4) = \max\ldots$$

Summary / Status

- Outlined a general frame-work for quantitative analysis
- including:
  - trace inclusion / equality
  - (Bi)simulation
- Results on mutually, topologically inequivalence, and
- A generalized result on the relationship of (bi)simulation and language inclusion/equality.
- (Decidability of accumulating simulation distance for WTA).
- A characterising WCTL logic, with d_+ trace semantics. (adequate and expressive).

Future work

- Duality of operators.
- A $d_{\bullet}$ wCTL path semantics.
- A $d_{\pm}$ wCTL path semantics.
- Maybe $d_{?}$ wCTL path semantics.
- Computability and Complexity (especially lower bounds).
- Continuity w.r.t composition.
- Game characterizations.
- ...and more.