Weighted Bisimulation Games

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Results

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Theorem

Bisimulation is polynomial time equivalent to safety games.

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Weignted Bisimulation is in $\mathrm{NP} \cap \mathrm{co} - \mathrm{NP}$.

Weighted Transition System

Definition

A weighted TS: states S, transitions $\rightarrow \subseteq S \times \Sigma \times \mathbb{R} \times S$

(graph or state machine, if you prefer)

Analysis

Logics, language, (bi)simulation.

Definition: Distances (values in $\mathbb{R} \cup \{\infty\}$)

point-wise	accumulating
$d_L^\bullet(\sigma, \tau) = \sup_i \lambda^i \lvert \sigma_i - \tau_i \rvert$	$d_L^+(\sigma, \tau) = \sum_i \lambda^i \lvert \sigma_i - \tau_i \rvert$

$\lambda \in [0, 1]$ is a fixed discounting factor.

Bisimulation

Definition: Bisimulation $\sim$

A relation $R \subseteq S \times S$ over $(S, \Sigma, \rightarrow)$ is a bisimulation relation provided that whenever $s \; R \; t$ and $\alpha \in \Sigma$, $c \in \mathbb{R}$ then:

- $s \xrightarrow{\alpha,c} s'$ implies, for some t', $t \xrightarrow{\alpha,c} t'$ and $s' \; R \; t'$,
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Definition: Weighted Bisimulation (Acc.)

A family of relations $\mathbf{R} = \{R_\epsilon \subseteq S \times S \mid \epsilon \geq 0\}$ is an (Acc.) bisim. family provided that for all $(s, t) \in R_\epsilon \in \mathbf{R}$:

- $s \xrightarrow{\alpha,c} s'$, implies $t \xrightarrow{\alpha,d} t'$ with $|c - d| \leq \epsilon$ for some $d \in \mathbb{R}_{\leq 0}$
- ...

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Definition: Weighted Bisimulation (Acc.)

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- $s \xrightarrow{\alpha,c} s'$, implies $t \xrightarrow{\alpha,d} t'$ with $|c - d| \leq \epsilon$ for some $d \in \mathbb{R}_{\leq 0}$ and $(s', t') \in R'_\epsilon \in \mathbf{R}$ with $\epsilon' \lambda \leq \epsilon' |c - d|$,
- ...

Example

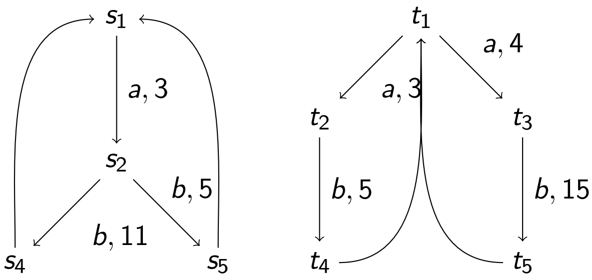

Figure: Example WTS

For which s_1 and t_1 are bisimilar and $s_1 \; R_{36.9} \; t_1$

Saftey Games

Definition

Given a graph $(V_1 \uplus V_2, E)$ where $E \subseteq V_i \times \Sigma \times V_{i+1 \mod 2}$. A safty game w.r.t a set $B \subseteq V_1$, invites players 1 and 2 to produce positional strategies which avoids indef. (resp. hits once) elements of B.

Saftey Games

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Define $W_B \subseteq S_1 \cup S_2$ as the vertices for which player 1 has a winning strategy avoiding B.

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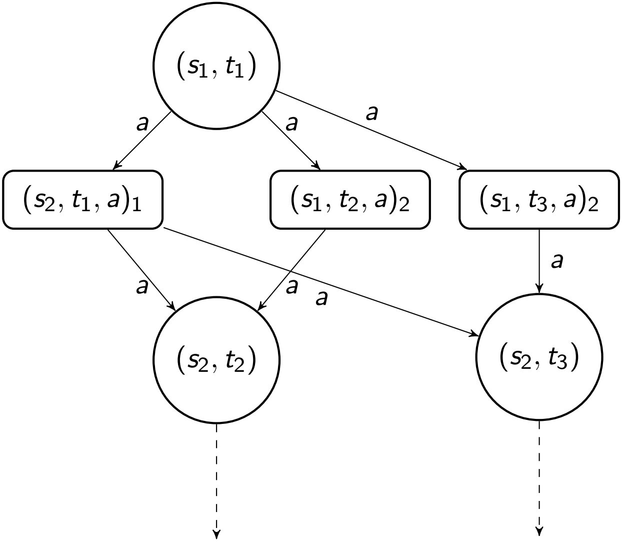
Definition

A (memory less) positional strategy for player i is a map $\sigma : S_i \to Act \times S_{i+1}$, consistent with E s.t. $\forall s \in S_i : \sigma(s_i) \in E(s_i)$.


(s_1, t_1)
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$(s_2, t_1, a)_1$
$(s_1, t_2, a)_2$
$(s_1, t_3, a)_2$
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(s_2, t_2)
(s_2, t_3)

Bisimulation $\rightsquigarrow$ Safty Game

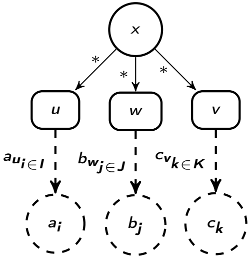
Given $(S, \Sigma, \rightarrowtriangle)$ construct the game $(V_1, V_2, A, \longrightarrow)$ such that:

- $V_2 = S \times S$ and $V_1 = S \times S \times Act \uplus S \times S \times Act$
- $(s,t) \xrightarrow{\star} (s', t, a)_1$ if $s \xrightarrow{a} s'$ and $(s,t) \xrightarrow{\star} (s, t', a)_2$ if $t \xrightarrow{a} t'$
- $(s,t,a)_1 \xrightarrow{\star} (s, t')$ if $t \xrightarrow{a} t'$ and $(s,t,a)_2 \xrightarrow{\star} (s', t)$ if $s \xrightarrow{a} s'$

Finally $B = \{(s,t,a)_1 \mid t \not\xrightarrow{a}\} \cup \{(s,t,a)_2 \mid s \not\xrightarrow{a}\}$

Theorem

Given states s and t of an LTS, then $s \sim t$ iff $(s,t) \in W_B$ of the corresponding game.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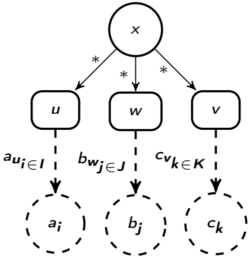
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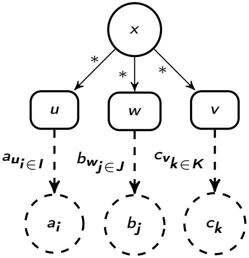
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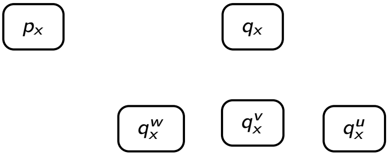
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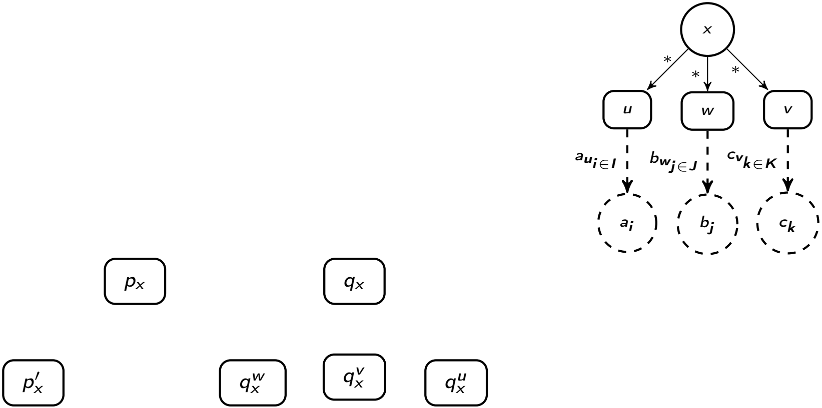
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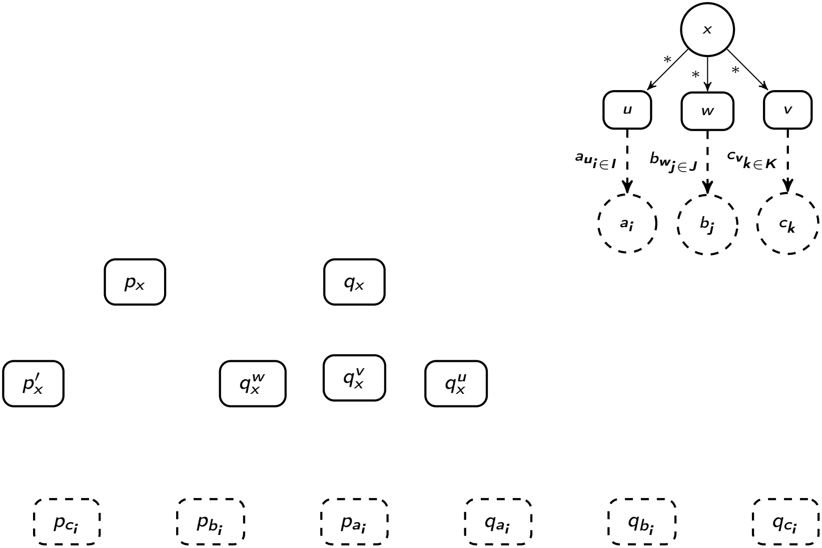
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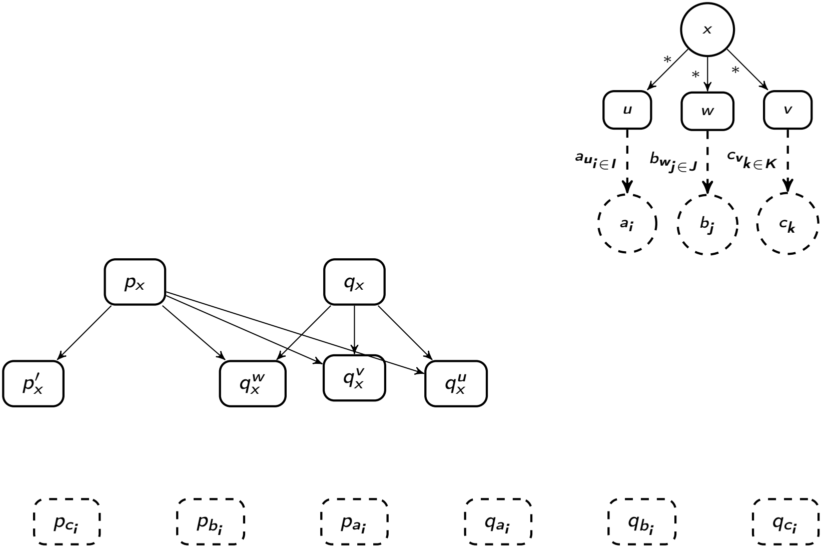

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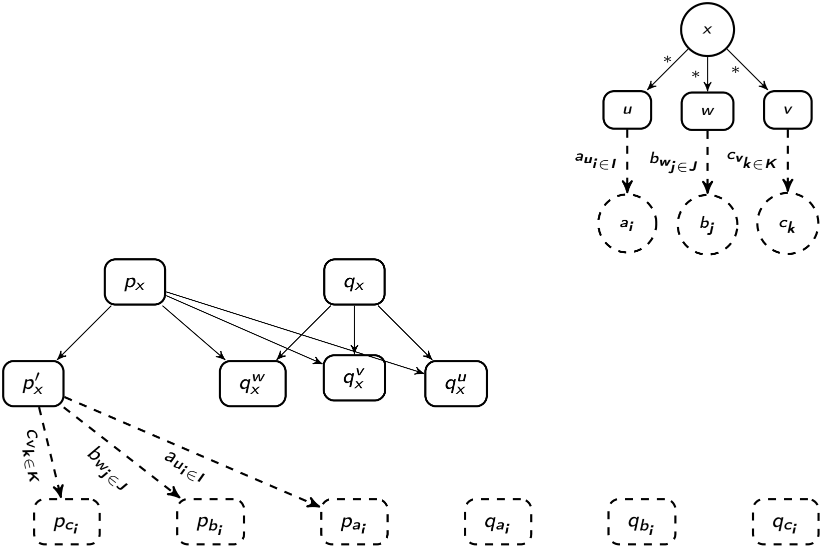

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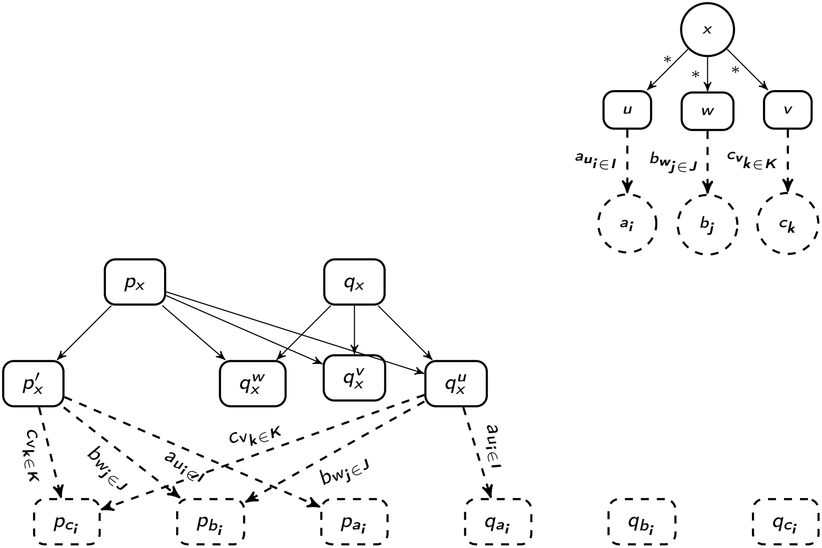
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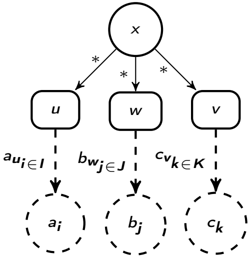
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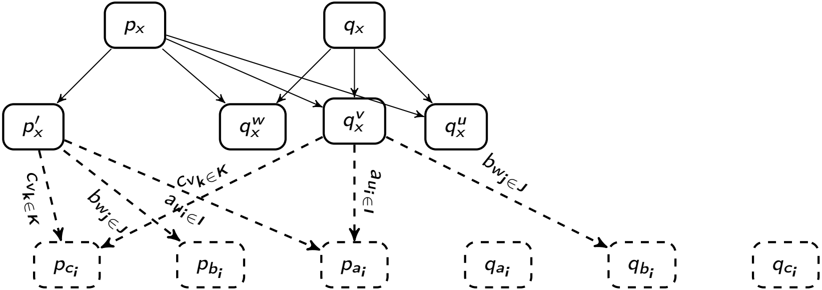
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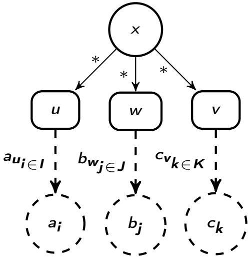
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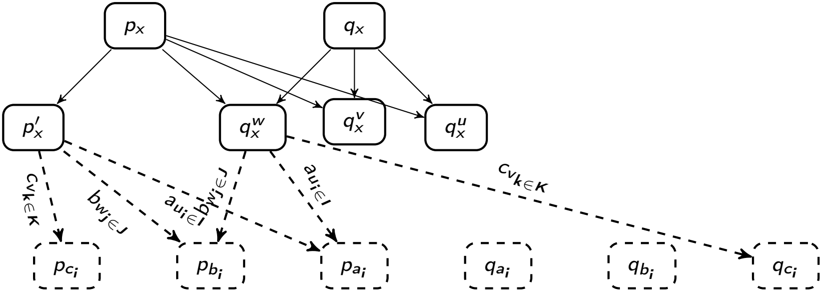
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q_{b_i}
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Safty Game $\rightsquigarrow$ Bisimulation

Given $(V_1, V_2, Act, \rightarrow)$ and $B \subseteq V_1$, construct $(S, \Sigma, \multimap)$ s.t:

- $S = \{p_x, p'_x, q_x \mid x \in V_1\} \cup \{q^u_x \mid x \in V_1, u \in V_2 \wedge \exists a : x \xrightarrow{a} u\} \cup \{\bot\}$
- $\Sigma = \{\tau\} \cup Act \times S_2$
- $\multimap$ contains $p_x \overset{b}{\multimap} \bot$ if $x \in B$, otherwise if $x \notin B$
  - $p_x \overset{\tau}{\multimap} p'_x$ whenever $x \in S_1$
  - $p_x \overset{\tau}{\multimap} q^u_x$ whenever $x \in S_1 \wedge \exists \beta . x \xrightarrow{\beta}_1 u$
  - $p'_x \overset{\alpha_u}{\multimap} p_a$ whenever $\exists \beta . x \xrightarrow{\beta}_1 u \wedge u \xrightarrow{\alpha}_2 a.$
  - $q_x \overset{\tau}{\multimap} q^u_x$ whenever $\exists \beta . x \xrightarrow{\beta} u$
  - $q^u_x \overset{\alpha_u}{\multimap} q_a$ whenever $\exists \beta . x \xrightarrow{\beta}_1 u \wedge u \xrightarrow{\alpha}_2 a$
  - $q^u_x \overset{\alpha_v}{\multimap} p_b$ whenever $\exists \beta . x \xrightarrow{\beta}_1 v \wedge v \xrightarrow{\alpha}_2 b$ for $(u \neq v)$

Theorem

Given vertices x and u of a game G, and states p_x, q_x, p'_x, q^u_x in the corresponding LTS constructed from G as above, it holds that:

- $p_x \sim q_x$ *iff* $x \in W_B$
- $p'_x \sim q^u_x$ *iff* $u \in W_B$

Proof

$\Leftarrow$ $E \triangleq \{(p_x, q_x) \mid x \in W_B\} \cup \{(p'_x, q^u_x) \mid u \in W_B \wedge \exists b.x \xrightarrow{\beta}_1 u\}$ is a bisimulation.

$\Rightarrow$ $W \triangleq \{x \mid p_x \sim q_x\} \cup \{u \mid p'_x \sim q^u_x\}$ is a post fixed-point of transformer:

$$\begin{aligned} \mathcal{W}(A) = &\{x \in S_1 \mid x \notin B \wedge \exists \beta \exists u \in A : x \xrightarrow{\beta}_1 u\} \cup \\ &\{u \in S_2 \mid \forall \beta \forall x : u \xrightarrow{\beta}_2 x \implies x \in A\} \end{aligned}$$

Discouted Games

Definition

Given a graph $(V_1 \uplus V_2, E)$ where $E \subseteq V_i \times \Sigma \times V_{i+1 \mod 2}$ and $W : E \to \mathbb{R}$. A discounted payoff game invites players 1 and 2 to produce positional strategies, maximixing (resp. minimizing) the accumulated (discounted) pay-off resulting from the infinite run induced by the respective strategies.

The value vector $\vec{x}$ — Zwick, Paterson

$$x_i = \begin{cases} \max_{x_i \xrightarrow{\alpha,c} x_j} \{c + \lambda x_j\} & s_j \in V_1 \\ \min_{x_i \xrightarrow{\alpha,c} x_j} \{c + \lambda x_j\} & s_j \in V_2 \end{cases}$$

Definition: Weighted Bisimulation (Acc.)

A family of relations $\mathbf{R} = \{R_\epsilon \subseteq S \times S \mid \epsilon \geq 0\}$ is an (Acc.) bisim. family provided that for all $(s, t) \in R_\epsilon \in \mathbf{R}$:

- $s \xrightarrow{\alpha,c} s'$, implies $t \xrightarrow{\alpha,d} t'$ with $|c - d| \leq \epsilon$ for some $d \in \mathbb{R}_{\leq 0}$ and $(s', t') \in R'_\epsilon \in \mathbf{R}$ with $\epsilon' \lambda \leq \epsilon' |c - d|$,
- ...

Definition: Branching distances are minimal fixed points

$$d_B^+(s,t) = \sup \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} |x - y| + \lambda d_B^+(s', t') \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} |x - y| + \lambda d_B^+(s', t') \end{cases}$$

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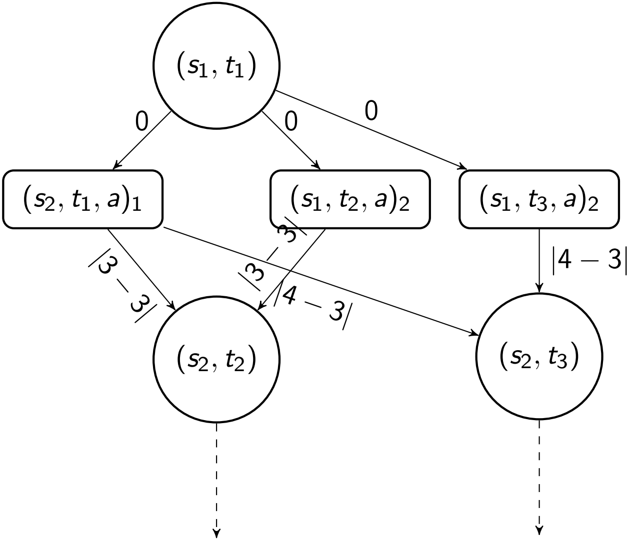
$$d_B^+(s,t) = \min\{\epsilon \mid s \; R_\epsilon \; t\}$$



(s_1, t_1)
0
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0
$(s_2, t_1, a)_1$
$(s_1, t_2, a)_2$
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$|3-3|$
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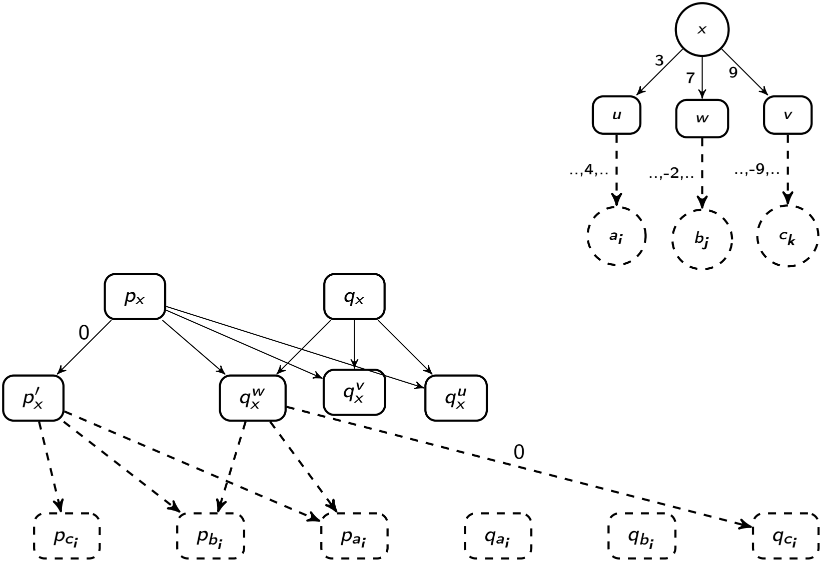
(s_2, t_2)
(s_2, t_3)

Theorem

Givens states s and t of a LTS, and vertex $x_{(s,t)}$ of the discounted game constructed as above.

$$d_B^+(s,t) = x_{(s,t)}$$

where d_B^+ is computed for d.f. λ and the game for $\sqrt{\lambda}$.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Theorem

Given a game vertex x, and the states p_x, q_x of the LTS constructed as above, then:

$$\vec{x}_x = d_B^+(p_x, q_x)$$

Conclusion

- Mean-payoff bisimulation.
- Point-wise bisimulation.
- maximum-lead bisimulation.