Computing Behavioral Distances, Compositionally

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Motivations

Markov Decision Processes with Rewards

- external nondeterminism + probabilistic behavior
- many useful applications (A.I., planning, games, biology, . . .)

Bisimilarity Distances

(bisimilarity is not robust: it only relates states with identical behaviors)

- measure the behavioral similarity between states
- support approximate reasoning on probabilistic systems
- need of efficient methods for computing bisim. distances

Compositionality $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n$

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  (the parallel composition of n systems with m states has m^n states!)
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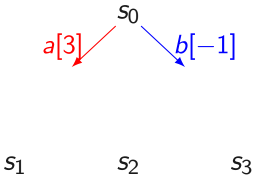

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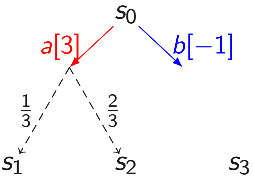

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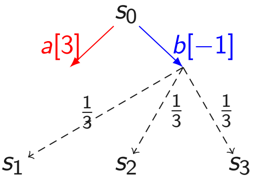

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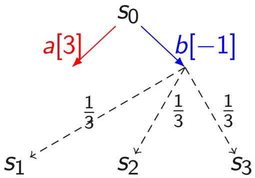

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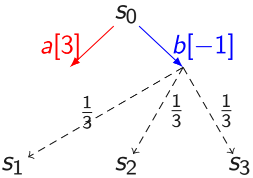

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Discounted accumulated reward $\lambda \in (0,1)$

$$R_\lambda(\omega) = \sum_{i\in\mathbb{N}} \lambda^i \cdot \rho(s_i, a_i)$$

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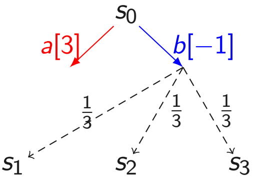

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Goal: To find policies $\pi \colon S \to A$ that maximize the expected value of R_λ on probabilistic executions starting from a given state.

Bisimilarity for MDPs

Extends probabilistic bisimilarity on Markov chains [Larsen-Skou'91]

Stochastic Bisimulation on $\mathcal{M}$ [Givan et al. AI'03]

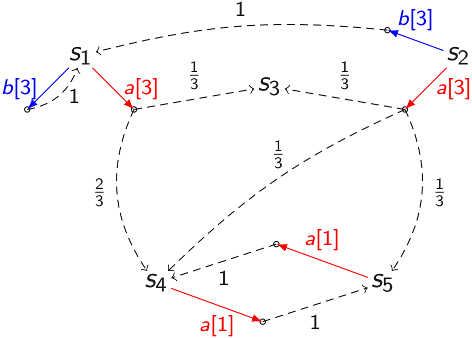
Equivalence relation $R \subseteq S \times S$ such that,

$$s \; R \; t \implies \forall a \in A. \begin{cases} \rho(s, a) = \rho(t, a) \\ \forall R\text{-equiv. class } C. \sum_{u \in C} \tau(s, a)(c) = \sum_{u \in C} \tau(t, v)(c) \end{cases}$$

Stochastic Bisimilarity on $\mathcal{M}$:

$$s \sim_{\mathcal{M}} t \iff s \; R \; t \text{ for some stochastic bisimulation } R \text{ on } \mathcal{M}.$$

Bisimilarity is not robust enough



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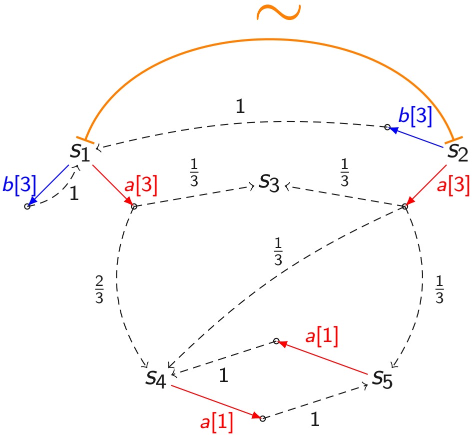

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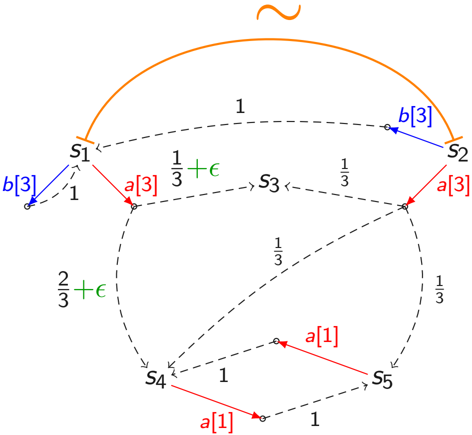

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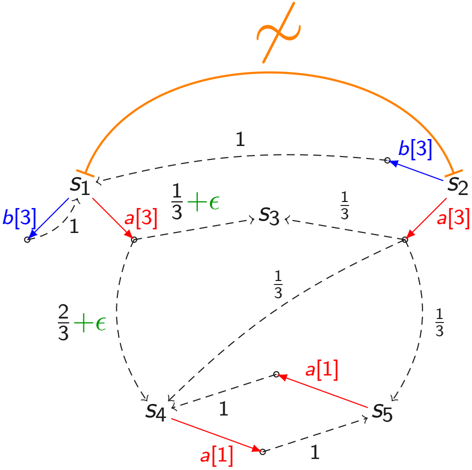

From equivalences to distances

Pseudometrics $d\colon S \times S \to \mathbb{R}_{\geq 0}$ are the quantitative analogue of an equivalence relation

equivalence		**pseudometric**
$s \equiv s$	$\rightsquigarrow$	$d(s,s) = 0$
$s \equiv t \implies t \equiv s$	$\rightsquigarrow$	$d(s,t) = d(t,s)$
$s \cong u \wedge u \cong t \implies s \cong t$	$\rightsquigarrow$	$d(s,u) + d(u,t) \geq d(s,t)$

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We consider the λ-discounted bisimilarity distances $\delta_\lambda\colon S \times S \to \mathbb{R}_{\geq 0}$ proposed by Ferns et al. [UAI'04]

From equivalences to distances

Bisimulation

$$\begin{tikzcd}
s \arrow[r, |-|, "R"] \arrow[d, mapsto] & t \arrow[d, mapsto] \\
\tau(s, a) \arrow[r, |-|, "\equiv_R"'] & \tau(t, a)
\end{tikzcd}$$

$\tau(s, a)(C) = \tau(t, a)(C)$

$\forall$ R-equiv. class

Metric analogue

$$\begin{tikzcd}
s \arrow[r, leftrightarrow, "d(s, t) < \epsilon"] \arrow[d, mapsto] & t \arrow[d, mapsto] \\
\tau(s, a) \arrow[r, leftrightarrow, "\mathcal{T}_d(\tau(s, a), \tau(t, a)) < \epsilon"'] & \tau(t, a)
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Kantorovich Metric: $\mathcal{T}_d \colon \Delta(S) \times \Delta(S) \to \mathbb{R}_{\geq 0}$

The distance between $\tau(s,a)$ and $\tau(t,a)$
is the optimal value of a Transportation Problem

$$\mathcal{T}_d(\tau(s,a),\tau(t,a)) = \min \left\{ \sum_{u,v \in S} d(u,v) \cdot \omega(u,v) \,\middle|\, \begin{array}{l} \forall u \in S \ \sum_{v \in S} \omega(u,v) = \tau(s,a)(u) \\ \forall v \in S \ \sum_{u \in S} \omega(u,v) = \tau(t,a)(v) \end{array} \right\}$$

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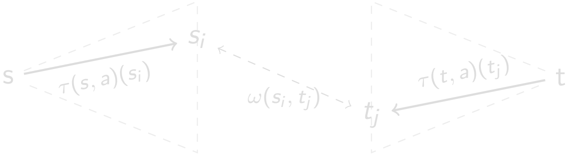

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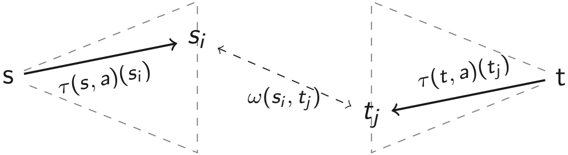

Bisimilarity Pseudometric [Ferns et al. UAI'04]

The bisimilarity pseudometric $\delta_\lambda^{\mathcal{M}}$ is the **least fixed point** of the following operator on pseudometrics

$$F_\lambda^{\mathcal{M}}(d)(s,t) = \max_{a \in A} \left\{ |\rho(s,a) - \rho(t,a)| + \lambda \cdot \mathcal{T}_d(\tau(s,a), \tau(t,a)) \right\}$$

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distance between rewards

and recursively. . .

distance between transition probabilities

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How to define operators on MDPs?

$$\mathcal{M}_1 \otimes \mathcal{M}_2 = \left(\underset{\substack{\uparrow \\ \text{set of} \\ \text{states}}}{S} , \underset{\substack{\uparrow \\ \text{set of} \\ \text{actions}}}{A} , \underset{\substack{\uparrow \\ \text{probability} \\ \text{transition} \\ \text{function}}}{\tau} , \underset{\substack{\uparrow \\ \text{reward} \\ \text{function}}}{\rho} \right)$$

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Example 1: Synchronous parallel composition

$$\mathcal{M}_1 \mid \mathcal{M}_2 = (S_1 \times S_2, A_1 \cap A_2, \tau_1 \mid_\tau \tau_2, \rho_1 \mid_\rho \rho_2)$$

$$\frac{s_1 \xrightarrow{a[r_1]}_{p_1} s_1' \qquad s_2 \xrightarrow{a[r_2]}_{p_2} s_2'}{s_1 \mid s_2 \xrightarrow{a[r_1+r_2]}_{p_1 \cdot p_2} s_1' \mid s_2'}$$

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$$(\rho_1 \mid_\rho \rho_2)((s_1, s_2), a) = \rho_1(s_1, a) + \rho_2(s_2, a)\,.$$

Example 2: CCS-like parallel composition

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Metric analogue of congruence

Operators over MDPs are well-behaved when they are congruencial w.r.t. bisimilarity:

$$s_1 \sim_{\mathcal{M}_1} t_1 \text{ and } s_2 \sim_{\mathcal{M}_2} t_2 \implies s_1 \otimes s_2 \sim_{\mathcal{M}_1 \otimes \mathcal{M}_2} t_1 \otimes t_2$$

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- + $\|\delta_\lambda^{\mathcal{M}_1}, \delta_\lambda^{\mathcal{M}_2}\|_1 \sqsupseteq \delta_\lambda^{\mathcal{M}_1 \otimes \mathcal{M}_2}$ ($\otimes$ is **non-extensive**)

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- + $\delta_\lambda^{\mathcal{M}_1}(s_1, t_1) + \delta_\lambda^{\mathcal{M}_2}(s_2, t_2) \geq \delta_\lambda^{\mathcal{M}_1 \otimes \mathcal{M}_2}(s_1 \otimes s_2, t_1 \otimes t_2)$
- + $\|\delta_\lambda^{\mathcal{M}_1}, \delta_\lambda^{\mathcal{M}_2}\|_p \sqsupseteq \delta_\lambda^{\mathcal{M}_1 \otimes \mathcal{M}_2}$ ($\otimes$ is p-**non-extensive**)

Safe algebraic operators on MDPs

Proving non-extensiveness for $\otimes$ may lead to rather involved proofs

($\delta_\lambda^{\mathcal{M}}$ is defined as the least fixed point of $F_\lambda^{\mathcal{M}}$)

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$$F_\lambda^{\mathcal{M}_1 \otimes \mathcal{M}_2}(\|d_1, d_2\|_p) \sqsubseteq \|F_\lambda^{\mathcal{M}_1}(d_1), F_\lambda^{\mathcal{M}_2}(d_2)\|_p$$

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Theorem: p-Safeness $\implies$ non-extensiveness

✓ Synch. parallel comp. ✓ CCS-like parallel comp.

Computing the behavioral distance

given $s, t \in S$, to compute $\delta_\lambda^{\mathcal{M}}(s, t)$

On-the-fly algorithm

[Bacci^2,Larsen,Mardare TACAS'13]

- lazy exploration of $\mathcal{M}$
- save comput. time $+$ space

Compositional strategy

- exploit the compositional structure of $\mathcal{M}_1 \otimes \mathcal{M}_2$

Alternative characterization of $\delta_\lambda^{\mathcal{M}}$

$$F_\lambda^{\mathcal{M}}(d)(s,t) = \max_{a \in A} \left\{ |\rho(s,a) - \rho(t,a)| + \lambda \cdot \mathcal{T}_d(\tau(s,a), \tau(t,a)) \right\}$$

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Coupling: $\mathcal{C} = \left\{ \omega_{s,t}^a \in \Pi(\tau(s,a), \tau(t,a)) \right\}_{s,t \in S}^{a \in A}$

$$\Gamma_\lambda^{\mathcal{C}}(d)(s,t) = \max_{a \in A} \left\{ |\rho(s,a) - \rho(t,a)| + \lambda \sum_{u,v \in S} d(u,v) \cdot \omega_{s,t}^a(u,v) \right\}$$

we call discrepancy, $\gamma_\lambda^{\mathcal{C}}$, the least fixed point of $\Gamma_\lambda^{\mathcal{C}}$

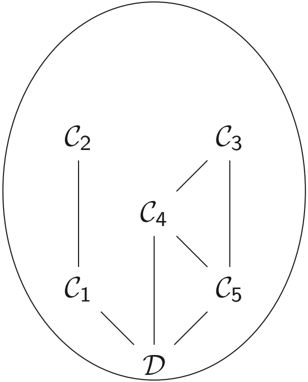
Theorem:

$\delta_\lambda^{\mathcal{M}} = \min\{\gamma_\lambda^{\mathcal{C}} \mid \mathcal{C} \text{ coupling for } \mathcal{M}\}$ for all $\lambda \in (0,1)$.

On-the-fly strategy

[Bacci^2-Larsen-Mardare TACAS'13]

$$\mathcal{C}_1 \trianglelefteq_\lambda \mathcal{C}_2 \iff \gamma_\lambda^{\mathcal{C}_1} \sqsubseteq \gamma_\lambda^{\mathcal{C}_2}$$



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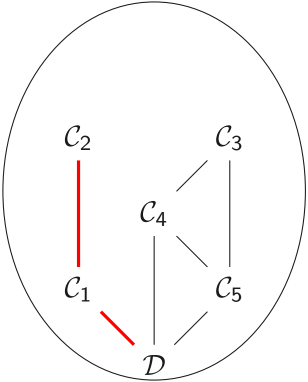

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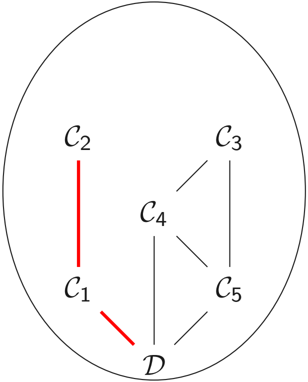

Greedy strategy

Moving Criterion:

$\mathcal{C}_i = \{\dots, \omega_{u,v}^a, \dots\}$

$\omega_{u,v}^a$ not opt. w.r.t. $TP(\gamma_\lambda^{\mathcal{C}_i}, \tau(u,a), \tau(v,a))$

Improvement:

$\mathcal{C}_{i+1} = \{\dots, \omega^*, \dots\}$, where

ω^* optimal sol. for $TP(\gamma_\lambda^{\mathcal{C}_i}, \tau(u,a), \tau(v,a))$

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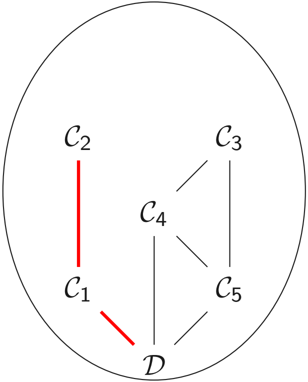

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Theorem

- each step ensures $\mathcal{C}_{i+1} \triangleleft_\lambda \mathcal{C}_i$
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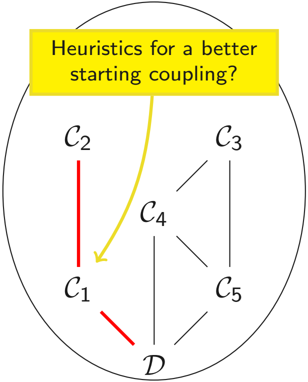
Heuristics for a better starting coupling?

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A Compositional Heuristic

How to obtain a good starting coupling $\mathcal{C}^*$?

Given that $\mathcal{M} = \mathcal{M}_2 \otimes \mathcal{M}_2$, non-extensiveness says that

$$\delta_\lambda^{\mathcal{M}_2 \otimes \mathcal{M}_2} \sqsubseteq \qquad\qquad \|\delta_\lambda^{\mathcal{M}_1}, \delta_\lambda^{\mathcal{M}_2}\|_p$$

Good?

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Good? when it doesn't exceed the upper-bound

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HOW? from $\mathcal{D}_1$ and $\mathcal{D}_2$

Lifting algebraic operators on Couplings

Lifting operator

$$\begin{tikzcd}
\mathcal{M}_1, \arrow[d, dashed] & \mathcal{M}_2 \arrow[r, mapsto] \arrow[d, dashed] & \mathcal{M}_1 \otimes \mathcal{M}_2 \arrow[d, dashed] \\
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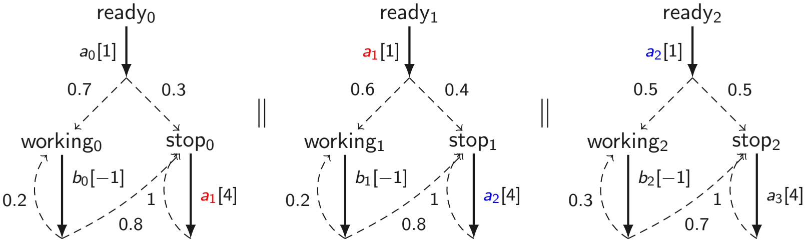
where $\mathcal{D}_i$ is a coupling for $\mathcal{M}_i$ minimal w.r.t. $\trianglelefteq_\lambda$

The Pipeline Example



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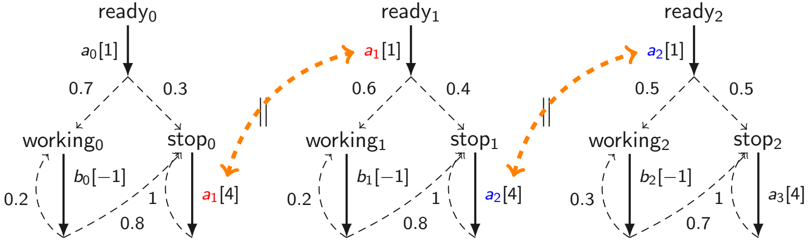

Experimental Results

Query	Instance	OTF	COTF	# States
All pairs	$E_0 \parallel E_1$	0.654791	0.97248	9
	$E_1 \parallel E_2$	0.702105	0.801121	9
	$E_0 \parallel E_0 \parallel E_1$	48.5982	13.5731	27
	$E_0 \parallel E_1 \parallel E_2$	23.1984	19.9137	27
	$E_0 \parallel E_1 \parallel E_1$	126.335	13.6483	27
	$E_0 \parallel E_0 \parallel E_0$	49.1167	14.1075	27
Single pair	$E_0 \parallel E_0 \parallel E_0 \parallel E_1 \parallel E_1$	16.7027	11.6919	243
	$E_0 \parallel E_1 \parallel E_0 \parallel E_1 \parallel E_1$	20.2666	16.6274	243
	$E_2 \parallel E_1 \parallel E_0 \parallel E_1 \parallel E_1$	22.8357	10.4844	243
	$E_1 \parallel E_2 \parallel E_0 \parallel E_0 \parallel E_2$	11.7968	6.76188	243
	$E_1 \parallel E_2 \parallel E_0 \parallel E_0 \parallel E_2 \parallel E_2$	Time-out	79.902	729

Conclusion and Future Work

Results

- generic definition of algebraic operators on MDPs
- characterized a well-behaved class of operators (p-Safeness)
- on-the-fly algorithm for behavioral pseudometrics
  - exact
  - avoids entire exploration of the state space
  - exploit compositional structure of the model **(first proposal!)**
- developed a proof of concept prototype [http://people.cs.aau.dk/giovbacci/tools.html]
- performs, on average, better than other proposals

Future work

- formal analysis of time/space complexity
- apply similar techniques on CTMCs, CTMDPs, etc...