On-the-Fly Exact Computation of Bisimilarity Distances

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Motivations

Probabilistic Systems

- lack of knowledge or inherent nondeterminism
- applied in various contexts (biology, security, games, A.I., ...)

Probabilistic Bisimulation is too fragile

- it only relates states with identical behaviors
- slight changes in quantities $\implies$ systems no more bisimilar

Bisimilarity Distances

- measure the behavioral similarity between states
- support approximate reasoning on probabilistic systems

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Markov Chains & Probabilistic Bisimilarity

Markov Chain: $\mathcal{M} = (S, \Sigma, \pi, \ell)$

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finite set of states

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set of labels

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probability transition function
$\pi : S \times S \to [0, 1]$ s.t. $\forall u \in S.\ \sum_{v \in S} \pi(u, v) = 1$

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labelling function
$\ell\colon S \to \Sigma$

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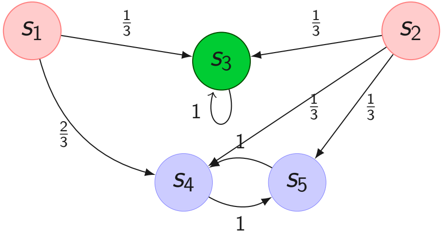

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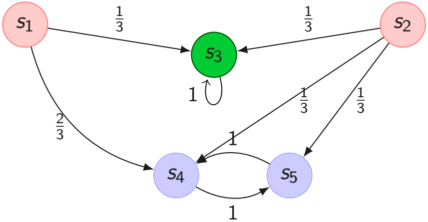
Probabilistic Bisimulation [Larsen and Skou]

Equivalence relation $R \subseteq S \times S$ such that

$$s \, R \, t \implies \begin{cases} \ell(s) = \ell(t) \\ \forall R\text{-equiv. class } C.\ \sum_{u \in C} \pi(s,u) = \sum_{v \in C} \pi(t,v) \end{cases}$$

Probabilistic Bisimilarity:

$s \sim t$ if $s \, R \, t$ for some probabilistic bisimulation R.



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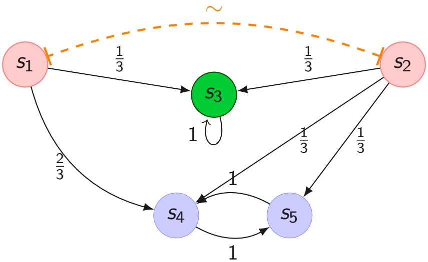

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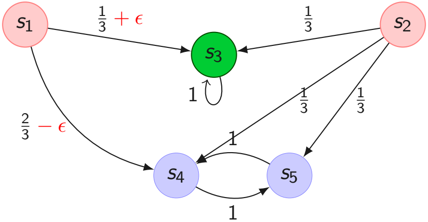

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FRAGILE

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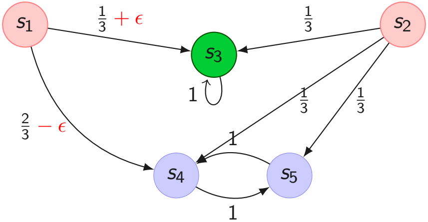


From equivalences to distances

Formalize distance as a pseudometric $d\colon S \times S \to [0,1]$

Quantitative analogue of an equivalence relation:

$$d(s,s) = 0\,, \quad d(s,t) = d(t,s)\,, \quad d(s,t) \leq d(s,u) + d(u,t)$$

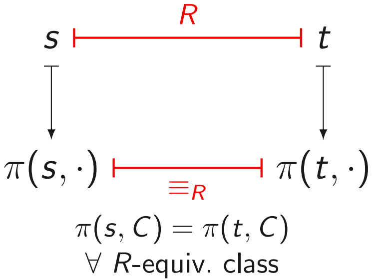
Bisimilarity Pseudometric: $d(s,t) = 0 \iff s \sim t$

We consider the λ-discounted bisimilarity distances $\delta_\lambda\colon S \times S \to [0,1]$ proposed by Desharnais et al.

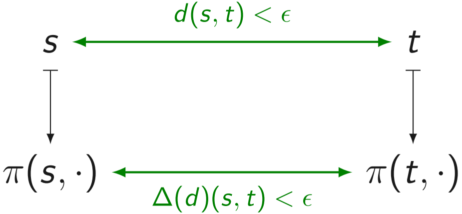
From equivalences to distances

. . . we directly introduce its fixed point characterization, given by van Breugel and Worrell

Bisimulation



Metric analogue



Kantorovich Metric

The distance between $\pi(s,\cdot)$ and $\pi(t,\cdot)$
is the optimal value of a Transportation Problem

$$\min \left\{ \sum_{u,v \in S} d(u,v) \cdot \omega(u,v) \;\middle|\; \begin{array}{l} \forall u \in S \ \sum_{v \in S} \omega(u,v) = \pi(s,u) \\ \forall v \in S \ \sum_{u \in S} \omega(u,v) = \pi(t,v) \end{array} \right\}$$

ω can be understood as transportation of $\pi(s,\cdot)$ to $\pi(t,\cdot)$.

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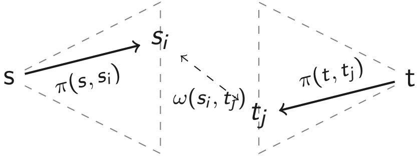

Bisimilarity Pseudometric [van Breugel, Worrel]

Given a parameter $\lambda \in (0, 1]$, called discount factor,
the bisimilarity pseudometric δ_λ is the **least fixed point** of

$$\Delta_\lambda(d)(s,t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \min\limits_{\omega \in \pi(s,\cdot) \otimes \pi(t,\cdot)} \lambda \sum\limits_{u,v \in S} d(u,v) \cdot \omega(u,v) & \text{otherwise} \end{cases}$$

Existing Methods for computing δ_λ

Iterative Method – Approximated:

$$\mathbf{0} \sqsubseteq \Delta_\lambda(0) \sqsubseteq \Delta_\lambda(\Delta_\lambda(0)) \cdots \sqsubseteq \Delta_\lambda^n(0) \sqsubseteq \cdots \sqsubseteq \delta_\lambda$$

Iterative Method – Exact: [Chen et al. – FoSSaCS'12]

+ compute an good approximation $\Delta_\lambda^n(0)$,
+ obtain an exact solution using the continued fraction algorithm

Linear programming: [Chen et al. – FoSSaCS'12]

+ solution of a linear program with exponentially many constraints
+ they provided a polynomial separation algorithm

$\Longrightarrow$ ellipsoid method

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On-the-fly approach

All existing methods requires to explore the entire state space

What if we only need $\delta_\lambda(s, t)$?

can we skip (part or all) the computation of some other pairs?

we propose an **on-the-fly** strategy:

- + lazy exploration of $\mathcal{M}$ (only where and when is needed)
- + saving in computational cost both in time and space

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Coupling: $\mathcal{C} = \{\omega_{s,t} \in \pi(s,\cdot) \otimes \pi(t,\cdot)\}_{s,t \in S}$

$$\begin{aligned} x_{s,t} &= 1 && \ell(s) \neq \ell(t) \\ x_{s,t} &= \lambda \textstyle\sum_{u,v \in S} x_{u,v} \cdot \omega_{s,t}(u,v) && \ell(s) = \ell(t) \end{aligned}$$

we call discrepancy, $\gamma_\lambda^{\mathcal{C}}$, the least solution of the linear system

Theorem: Chen, van Breugel and Worrel – FoSSaCS'12

$\delta_1 = \min\{\gamma_1^{\mathcal{C}} \mid \mathcal{C} \text{ coupling for } \mathcal{M}\}$

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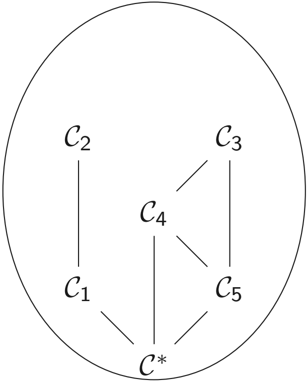
On-the-fly strategy

$$\mathcal{C}_1 \trianglelefteq_\lambda \mathcal{C}_2 \iff \gamma_\lambda^{\mathcal{C}_1} \sqsubseteq \gamma_\lambda^{\mathcal{C}_2}$$



Greedy strategy

Moving Criterion:

$\mathcal{C}_i = \{\dots, \omega_{u,v}, \dots\}$

$\omega_{u,v}$ not opt. w.r.t. $TP(\gamma_\lambda^{\mathcal{C}_i}, \pi(u,\cdot), \pi(v,\cdot))$

Improvement:

$\mathcal{C}_{i+1} = \{\dots, \omega^*, \dots\}$, where

ω^* optimal sol. for $TP(\gamma_\lambda^{\mathcal{C}_i}, \pi(u,\cdot), \pi(v,\cdot))$

Theorem

- each step ensures $\mathcal{C}_{i+1} \triangleleft_\lambda \mathcal{C}_i$
- moving criterion holds until $\gamma_\lambda^{\mathcal{C}_i} \neq \delta_\lambda$
- the method always terminates

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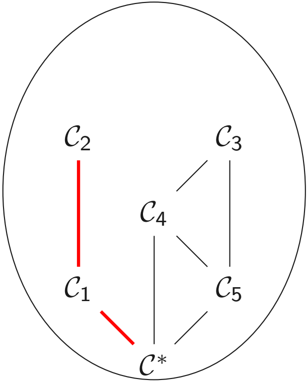

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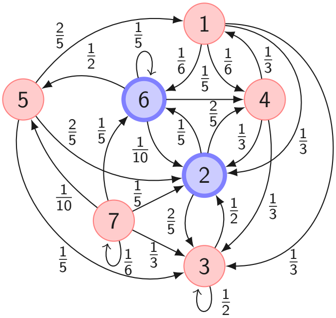
Goal: compute $\delta_1(1, 4)$



Solution:

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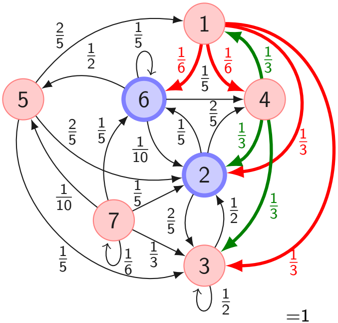

(1,4)	1	2	3	
2	$\frac{1}{3}$			$\frac{1}{3}$
3		$\frac{1}{3}$		$\frac{1}{3}$
4			$\frac{1}{6}$	$\frac{1}{6}$
6			$\frac{1}{6}$	$\frac{1}{6}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$
\begin{aligned}
d(1,4) &= \frac{1}{3} \cdot \overbrace{d(1,2)}^{=1} + \frac{1}{3} \cdot \overbrace{d(2,3)}^{=1} + \frac{1}{6} \cdot d(3,4) + \frac{1}{6} \cdot \overbrace{d(3,6)}^{=1} \\
&= \frac{1}{6} \cdot d(3,4) + \frac{5}{6}
\end{aligned}
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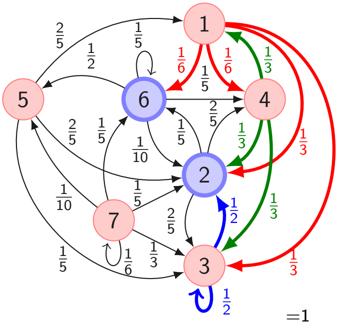

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$$= \frac{1}{2}$$

Solution:

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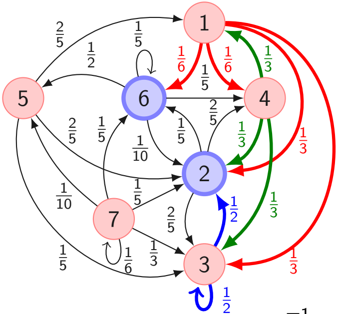

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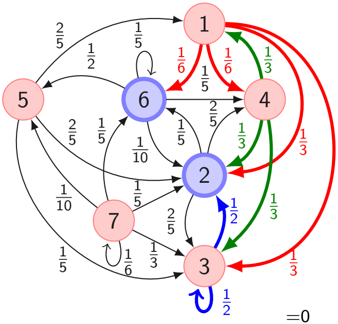
Solution: $d(1,4) = \frac{11}{12}$ and $d(3,4) = \frac{1}{2}$

Goal: compute $\delta_1(1,4)$



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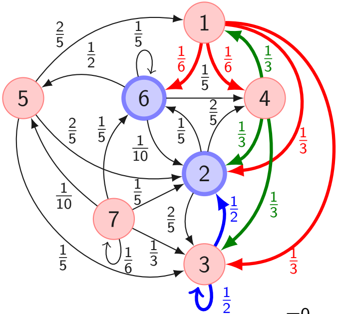
4	$\frac{1}{6}$			$\frac{1}{6}$
6	$\frac{1}{6}$			$\frac{1}{6}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$
\begin{aligned}
d(1,4) &= \frac{1}{3} \cdot \overbrace{d(2,2)}^{=0} + \frac{1}{3} \cdot \overbrace{d(3,3)}^{=0} + \frac{1}{6} \cdot d(1,4) + \frac{1}{6} \cdot \overbrace{d(1,6)}^{=1} \\
&= \frac{1}{6} \cdot d(1,4) + \frac{1}{6}
\end{aligned}
$$

Solution:

Goal: compute $\delta_1(1,4)$



(1,4)	1	2	3	
2		$\frac{1}{3}$		$\frac{1}{3}$
3			$\frac{1}{3}$	$\frac{1}{3}$
4	$\frac{1}{6}$			$\frac{1}{6}$
6	$\frac{1}{6}$			$\frac{1}{6}$
	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	

$$\begin{aligned} d(1,4) &= \frac{1}{3}\cdot\overbrace{d(2,2)}^{=0}+\frac{1}{3}\cdot\overbrace{d(3,3)}^{=0}+\frac{1}{6}\cdot d(1,4)+\frac{1}{6}\cdot\overbrace{d(1,6)}^{=1} \\ &= \frac{1}{6}\cdot d(1,4)+\frac{1}{6} \end{aligned}$$

Solution: $d(1,4) = \frac{1}{5}$

Empirical Results (all-pairs)

# States	On-the-Fly (exact)		Iterating (approximated)			Approx. Error*
	Time (s)	# TPs	Time (s)	# Iterations	# TPs	
5	0.019	1.191	0.0389	1.733	26.733	0.139
6	0.059	3.046	0.092	1.826	38.133	0.146
7	0.138	6.011	0.204	2.194	61.728	0.122
8	0.255	8.561	0.364	2.304	83.028	0.117
9	0.499	12.042	0.673	2.579	114.729	0.111
10	1.003	18.733	1.272	3.111	174.363	0.094
11	2.159	25.973	2.661	3.556	239.557	0.096
12	4.642	34.797	5.522	4.042	318.606	0.086
13	6.735	39.958	8.061	4.633	421.675	0.097
14	6.336	38.005	7.188	4.914	593.981	0.118
17	11.261	47.014	12.805	5.885	908.61	0.132
19	26.635	61.171	29.654	6.961	1328.60	0.140
20	34.379	66.457	38.206	7.538	1597.92	0.142

(*) $\epsilon = \max_{s,t \in S} \delta_\lambda(s,t) - d(s,t)$

Empirical Results (single-pair)

# States	out-degree $= 3$		$2 \leq$ out-degree $\leq$ # States	
	Time (s)	# TPs	Time (s)	# TPs
5	0.006	0.273	0.012	0.657
6	0.012	0.549	0.031	1.667
7	0.017	0.981	0.088	3.677
8	0.025	1.346	0.164	5.301
9	0.026	1.291	0.394	8.169
10	0.058	2.038	1.112	13.096
11	0.077	1.827	2.220	18.723
12	0.043	1.620	4.940	26.096
13	0.060	1.882	10.360	35.174
14	0.089	2.794	20.123	46.077

Conclusion and Future Work

Results

- on-the-fly algorithm for bisimulation metrics
  - exact
  - avoids entire exploration of the state space
- we developed a proof of concept prototype
- performs, on average, better than other proposals

Future work

- formal analysis of time/space complexity
- apply similar techniques in other contexts (e.g. MDPs, CTMCs, CTMDPs)
- exploit compositional structure of the model