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Propositional Logics for the Lawvere Quantale

Giorgio Bacci[a,1] Radu Mardare[b,2] Prakash Panangaden[c,3] Gordon Plotkin[d,4]

[a] *Department of Computer Science, Aalborg University, Aalborg, Denmark*

[b] *Computer and Information Sciences, University of Strathclyde, Glasgow, Scotland*

[c] *School of Computer Science, McGill University and Mila, Montréal, Canada*

[d] *LFCS, School of Informatics, University of Edinburgh, Edinburgh, Scotland*

Abstract

Lawvere showed that generalised metric spaces are categories enriched over $[0, \infty]$, the quantale of the positive extended reals. The statement of enrichment is a quantitative analogue of being a preorder. Towards seeking a logic for quantitative metric reasoning, we investigate three $[0, \infty]$-valued propositional logics over the Lawvere quantale. The basic logical connectives shared by all three logics are those that can be interpreted in any quantale, viz finite conjunctions and disjunctions, tensor (addition for the Lawvere quantale) and linear implication (here a truncated subtraction); to these we add, in turn, the constant $\mathbb{1}$ to express integer values, and scalar multiplication by a non-negative real to express general affine combinations. Quantitative equational logic can be interpreted in the third logic if we allow inference systems instead of axiomatic systems.

For each of these logics we develop a natural deduction system which we prove to be decidably complete w.r.t. the quantale-valued semantics. The heart of the completeness proof makes use of the Motzkin transposition theorem. Consistency is also decidable; the proof makes use of Fourier-Motzkin elimination of linear inequalities. Our logics are variants on $[0, 1]$-logics, via the exponential map e^{-x}: the first is equivalent to Goguen's product logic [9], the second is a version of product logic with a fixed choice of constants, the third adds exponentiation. The first is known; the second and third are apparently novel. The third is natural in the additive $[0, \infty]$-context, but not the multiplicative $[0, 1]$-context. Our proofs are novel in all cases, making uses of linear algebraic results; these results are commonly regarded informally as a kind of completeness result; here we put that view to good use. Strong completeness does not hold in general, even (as is known) for theories over finitely-many propositional variables; indeed even an approximate form of strong completeness in the sense of Pavelka or Ben Yaacov—provability up to arbitrary precision—does not hold. However, we can show it for theories axiomatized by a (not necessarily finite) set of judgements in *normal form* over a finite set of propositional variables when we restrict to models that do not map variables to ∞; the proof uses Hurwicz's general form of the Farkas' Lemma.

Keywords: Quantitative reasoning, axiomatizations, quantitative algebras, metric spaces, quantale-valued logics.

1 Introduction

Real-valued logics have been experiencing a recent resurgence of interest because of probabilistic [1, 15, 16] and metric reasoning [24] and applications such as neurosymbolic reasoning in machine learning [5, 6, 28]. They are generally fuzzy logics [12] often interpreted over $[0, 1]$—for example Łukasiewicz logic ([19, 33, 34]). In [18], Lawvere showed that generalised metric spaces are categories enriched over $[0, \infty]$, the quantale of the positive extended reals. The statement of enrichment is a quantitative analogue of being a preorder. One can view Łukasiewicz logic

[1] Email: grbacci@cs.aau.dk
[2] Email: r.mardare@strath.ac.uk
[3] Email: prakash@cs.mcgill.ca
[4] Email: gdp@inf.ed.ac.uk

as the propositional logic of a quantale over $[0,1]$, see [3]. Also there are interpretations of (variants of) linear logic in quantales [35].

In this paper, we propose studying logic over Lawvere's quantale, and we begin such a study with propositional logic. An argument that points towards this quantale comes also from the literature on quantitative algebras [20], where the development of quantitative equational logic is based on a family of binary predicates "$=_\varepsilon$" for every $\varepsilon \geq 0$. These are used in quantitative equations, such as $s =_\varepsilon t$, to encode the fact that the distance between the interpretation of the terms s and t in a metric space is at most ε. Once this quantitative information is embedded in the predicate "$=_\varepsilon$", quantitative equations become Boolean statements, i.e., *true* or *false* in a model. An alternative way to tackle this issue is to consider only one "$=$" predicate that is valued in the Lawvere quantale. This requires exchanging the classical "Boolean core" of quantitative equational logic with a many-valued logic interpreted over Lawvere's quantale.

In this paper, we investigate basic concepts and proof systems for such propositional logics. We consider three closely related logics, built up in stages. The basic logical connectives, shared by all three logics, are those that can be interpreted in any quantale, viz finite conjunctions and disjunctions, tensor (addition for the Lawvere quantale) and linear implication (here a truncated subtraction). To these we add, in turn, the constant $\mathbb{1}$ to express integer values, and scalar multiplication by a non-negative real to express general non-negative affine combinations. Quantitative equational logic can be interpreted in the third logic once we extend the provability principles and instead of only looking to theories defined by axiomatic systems, we also consider theories closed under systems of inferences.

Our logics are variants of $[0,1]$-valued logics, via the exponential map e^{-x}: the first is equivalent to product logic [11, 13], the second is a version of product logic with a fixed choice of constants [10, 26], the third adds exponentiation. The first is known; the second and third are apparently novel. The third is natural in the additive $[0,\infty]$-context, but not the multiplicative $[0,1]$-context, where multiplication by a constant becomes exponentiation. Multiplication by a constant can be viewed as a graded modality; however the literature on product logic modalities seems to consider only Kripke models [31].

Our proofs are novel in all cases. The proofs used in the fuzzy logic literature, e.g., in [13, 26], employ the methods of algebraic logic, and for product logic in particular, use results of ordered Abelian group theory. Our proofs are arguably simpler. They consist of a syntactic reduction to a normal form using only affine combinations of propositional variables, followed by a novel application of theorems from linear algebra, such as the Farkas' Lemma. It has been long noted in the literature [22] that the Farkas' Lemma, and related results, can be thought of as completeness theorems. Here they are literally seen as such, and used to prove general completeness for our propositional logics.

For each of these logics we develop a natural deduction system, and prove it complete relative to interpretations in the Lawvere quantale. The proof of completeness uses a normalisation technique which replaces a sequent $\varphi \vdash \psi$ that is not in normal form, with a finite set of sequents in normal form. The main normal form is a sequent where the formulas φ, ψ are tensors $r_1 * p_1 \otimes \ldots \otimes r_n * p_n \otimes r$ of propositional variables multiplied by positive reals, and a scalar. Semantically, these are exactly affine linear combinations of the variables. Leaving aside the details, this reduces the problem of proving that a given sequent follows from a given finite set of sequents, to the problem of proving that a given affine inequality is a consequence of a given finite set of affine inequalities. This is exactly the province of (variants of) the Farkas' Lemma [7] and the Motzkin transposition theorem [23]. These show that when such a consequence holds for all values of the variables, then the given affine inequality is a linear combination of the given set of inequalities (an integer variant of Motzkin [30] is used when the reals are integers). It is in this way that the Farkas' Lemma and its relatives can be viewed as completeness results (rather than, as in another–very similar–view, as a dichotomy result that either there is a linear combination or there is a counterexample).

As reduction to sets of normal forms and the Farkas' Lemma (and related) are effective, it follows that satisfiability is decidable. We can also decide consistency via the reduction to normal form, now making use of Fourier-Motzkin elimination [8, 32]. By making these reductions efficient one can further show that consequence is co-NP complete and that consistency is NP-complete.

Strong completeness does not hold in general, even for theories over finitely-many propositional variables; indeed even an approximate form of strong completeness in the sense of Pavelka or Ben Yaacov [4, 25, 33, 34]—provability up to arbitrary precision—does not hold. However, we can show it does hold for such theories that admit a, not necessarily finite, set of axioms in *normal form* over a *finite* set of propositional variables when we restrict to models that do not map variables to ∞; the proof uses Hurwicz's general form [14] of the Farkas' Lemma.

Synopsis
In Section 2, we present some preliminaries and basic notation. In Section 3, we introduce the three quantale-valued logics with their syntax and semantics. Section 4 is dedicated to natural deduction systems, one for each of the three logics. It contains soundness statements and a series of results regarding provability in these logics. A subsection here is dedicated to deduction theorems, with a detailed discussion about the failure of the classic deduction theorem. In Section 5, we collect a series of model-theoretic results and we introduce the concept of a diagrammatic theory that is central in the model theory of these logics. Section 6 is dedicated to normalization; it contains a brief description of an algorithm that computes the normal form of a sequent. Due to space restrictions, the algorithm is only explained for a couple of key cases. Section 7 is dedicated to completeness and incompleteness results for all these logics. In Section 8, we show how we can encode quantitative equational logic in our logic. Quantitative equational reasoning requires more than just an axiomatic system; it needs a system of inferences that does not produce one axiomatic theory, but a set of axiomatic theories.

2 Preliminaries and Notation

A *quantale* is a complete lattice with a binary, associative operation $\otimes$ (the *tensor*) preserving all joins (and so with both $a \otimes -$ and $- \otimes a$ having adjoints). A quantale is called *commutative* whenever its tensor is; it is called *unital* if there is an element 1, the unit, such that $1 \otimes a = a = a \otimes 1$, for all a; and it is called *integral* if the unit is the top element. For commutative quantales, where the adjoints coincide, we denote the adjoint to $a \otimes -$ by $a \multimap -$, which is characterised by

$$a \otimes b \leq c \Longleftrightarrow b \leq a \multimap c\,.$$

The complete lattice $[0, \infty]$ ordered by the "greater or equal" relation $\geq$ with extended sum as tensor, is known as the *Lawvere quantale* (or metric quantale). In this paper, we mainly work with the Lawvere quantale, so it is convenient to have an explicit characterisation of its basic operations. Join and meet are $\inf$ and $\sup$, respectively, ∞ is the bottom element and 0 the top. For $r, s \in [0, \infty]$, we define truncated subtraction as

$$r \dot{-} s = \begin{cases} 0 & \text{if } r \leq s \\ r - s & \text{if } r > s \text{ and } r \neq \infty \\ \infty & \text{if } r = \infty \text{ and } s \neq \infty\,. \end{cases}$$

Then, the right adjoint $s \multimap r$ is just $r \dot{-} s$ (note that the order of terms is inverted). We remark that the continuous t-norms of fuzzy logic are exactly the integral commutative quantales over $[0, 1]$, except that the condition of preserving all sups is replaced by the stronger condition of monotonicity plus continuity (in the usual sense) [12].

3 Logics for the Lawvere Quantale

In this section, we present three propositional logics interpreted over the Lawvere quantale which we will collectively refer to as *logics for the Lawvere quantale* (LLQ).

Syntax of logical formulas. Formulas are freely generated from a set $\mathbb{P} = \{p_1, p_2, \dots\}$ of atomic propositions over logical connectives that can be interpreted in the Lawvere quantale:

$$\begin{array}{lr} \bot \mid \top \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \otimes \psi \mid \phi \multimap \psi & \text{(quantale connect.)} \\ \mathbb{1} & \text{(constant)} \\ r * \phi \quad (\text{for } r \in [0, \infty)) & \text{(scalar multiplication)} \end{array}$$

The first logic, $\mathbb{L}$, uses only the basic logical connectives that can be interpreted in any commutative quantale, viz, the constants bottom ($\bot$) and top ($\top$), binary conjunction ($\wedge$) and disjunction ($\vee$), tensor ($\otimes$), and linear implication ($\multimap$).

The second logic, $\mathbb{L}_{\mathbb{1}}$, additionally allows the use of the constant $\mathbb{1}$.

The third logic, $\mathbb{L}_{\mathbb{1}}^*$, extends the syntax further with scalar multiplication by a non-negative real ($r * -$).

It will be useful to define, in all LLQ, the following derived connectives:

$$\neg\phi := \phi \multimap \bot\,, \qquad \text{(Negation)}$$
$$\phi \multimap\!\!\circ\, \psi := (\phi \multimap \psi) \wedge (\psi \multimap \phi)\,. \qquad \text{(Double implication)}$$

Moreover, for any $n \in \mathbb{N}$, the derived connective $n\phi$ is inductively defined as follows

$$0\phi := \top \qquad \text{and} \qquad (n+1)\phi := \phi \otimes n\phi\,.$$

Thus $\mathbb{L}_{\mathbb{1}}$ effectively has all natural numbers as constants via the formulas $r * \mathbb{1}$. (In fact $\mathbb{L}_{\mathbb{1}}$ is equivalent to the logic obtained from $\mathbb{L}$ by adding all rationals as constants.) Similarly, for any $r \in [0, \infty)$, we write simply r to denote the formula $r * \mathbb{1}$.

Notation 3.1 *To simplify the presentation, we assume an operator precedence rule so that $*$ binds most strongly, followed by $\otimes$, next are $\wedge$ and $\vee$, and the weakest are $\multimap$, $\multimap\!\!\circ$ and $\neg$. Thus, the formula $r * \phi \otimes \psi \wedge s * \psi \multimap \theta$ is interpreted as $(((r * \phi) \otimes \psi) \wedge (s * \psi)) \multimap \theta$.*

Semantics of logical formulas. The models of LLQ are maps $m \colon \mathbb{P} \to [0, \infty]$ interpreting the propositional symbols in the Lawvere quantale, which can be extended uniquely to formulas by setting

$$\begin{aligned} m(\bot) &:= \infty\,, & m(\phi \wedge \psi) &:= \max\{m(\psi), m(\phi)\}\,, \\ m(\top) &:= 0\,, & m(\phi \vee \psi) &:= \min\{m(\psi), m(\phi)\}\,, \\ m(\mathbb{1}) &:= 1\,, & m(\phi \otimes \psi) &:= m(\phi) + m(\psi)\,, \\ m(r * \phi) &:= r m(\phi)\,, & m(\phi \multimap \psi) &:= m(\psi) \dot{-} m(\phi)\,, \end{aligned}$$

with derived connectives $\neg$ and $\multimap\!\!\circ$ interpreted as

$$m(\neg\phi) = \infty \dot{-} m(\phi)\,, \qquad m(\phi \multimap\!\!\circ\, \psi) = |m(\psi) - m(\phi)|\,.$$

4 Natural Deduction Systems

We present natural deduction systems for the thee logics $\mathbb{L} \subseteq \mathbb{L}_{\mathbb{1}} \subseteq \mathbb{L}^*_{\mathbb{1}}$. As each logic is intended to be a conservative extension of its sub-logics, we present their deduction systems incrementally.

Let $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}, \mathbb{L}^*_{\mathbb{1}}\}$. A *judgement* in $\mathcal{L}$ is a syntactic construct of the form

$$\phi_1, \ldots, \phi_n \vdash \psi\,, \qquad \text{(Judgement)}$$

where the ϕ_i and ψ are logical formulas of $\mathcal{L}$, called respectively the *antecedents* and the *consequent* of the judgement. Note that the antecedents $\Gamma = (\phi_1, \ldots, \phi_n)$ of a judgement form a finite ordered list, possibly, with repetitions. As is customary, for Γ and Δ lists of formulas, their comma-separated juxtaposition Γ, Δ denotes concatenation; and $\vdash \psi$ is the notation for a judgement with empty list of antecedents.

A judgement $\gamma = (\Gamma \vdash \psi)$ in $\mathcal{L}$ *is satisfied by* a model m, in symbols $m \models_{\mathcal{L}} \gamma$, whenever

$$\textstyle\sum_{\phi \in \Gamma} m(\phi) \geq m(\psi)\,. \qquad \text{(Semantics of judgements)}$$

When the logic $\mathcal{L}$ is clear from the context or the satisfiability holds in all LLQ, we simply write $m \models \gamma$.

A judgement is *satisfiable* if it is satisfied by a model; *unsatisfiable* if it is not satisfiable; and a *tautology* if it is satisfied by all models. Note that, for any model m in LLQ

$$\begin{aligned} m &\models (\vdash \phi) && \text{iff} \quad m(\phi) = 0 \\ m &\models (\vdash \neg\phi) && \text{iff} \quad m(\phi) = \infty \quad (\textit{i.e.}, \phi \text{ is infinite}) \\ m &\models (\vdash \neg\neg\phi) && \text{iff} \quad m(\phi) < \infty \quad (\textit{i.e.}, \phi \text{ is finite}) \\ m &\models (\phi \vdash \psi) && \text{iff} \quad m(\phi) \geq m(\psi)\,. \end{aligned}$$

In particular, $\vdash \phi \multimap \phi$, $\vdash \top$, and $\vdash \neg\bot$ are examples of tautologies, while $\vdash \phi \multimapboth (\neg\neg\phi)$ is not. Moreover, by using negation we can express whether the interpretation of a formula is either finite or infinite.

An *inference (rule)* is a syntactic construct of the form $\dfrac{S}{\gamma}$, for S a set of judgements and γ a judgement. The judgements in S are the *hypotheses of the inference* and γ is the *conclusion of the inference*. When $S = \{\gamma'\}$ is a singleton, we write

$$\frac{\gamma'}{\overline{\gamma}} \quad \text{to denote that both} \quad \frac{\gamma'}{\gamma} \quad \text{and} \quad \frac{\gamma}{\gamma'} \quad \text{hold.}$$

A judgement γ is a *semantic consequence* of a set S of judgements, in symbols $S \models \gamma$, if every model that satisfies all the judgements in S satisfies also γ. Thus, $\emptyset \models \gamma$ (or more simply $\models \gamma$) means that γ is a tautology. For a model m, we will also use the notation $S \models_m \gamma$, to mean that, whenever m satisfies all the judgements of S, then it satisfies also γ.

The natural deduction system of $\mathbb{L}$

It consists of the inference rules in Figure 1. Figure 1a contains the basic rules of logical deduction (ID) and (CUT), and the structural rules of weakening (WEAK) and permutation (PERM). Note that, there is no cancellation rule. Figure 1b provides the rules for the lattice operations of the Lawvere quantale [5]. Figure 1c collects the rules that are specific to the Lawvere quantale. (WEM) is the weak excluded middle; (TOT) states that the quantale is totally ordered; the other rules explain the actions of $\otimes$ and its adjoint in the Lawvere quantale.

The natural deduction system of $\mathbb{L}_{\mathbb{1}}$

It includes all the rules in Figure 1 and, in addition,

$$\frac{\vdash \mathbb{1} \vee \neg\mathbb{1}}{\vdash \bot} \ (\text{ONE})$$

expressing that $0 \geq 1$ and $1 \geq \infty$ are inconsistencies. Thus, with $\mathbb{L}_{\mathbb{1}}$ we exit the universe of classical logic, as $\mathbb{1}$ cannot be provably equivalent either to $\top$, or to $\bot$.

The natural deduction system of $\mathbb{L}^*_{\mathbb{1}}$

It extends the deduction system of $\mathbb{L}_{\mathbb{1}}$ with the rules for scalar multiplication in Figure 2. In (S_4), $\bowtie$ can be either of $\wedge, \vee, \otimes, \multimap$, meaning that we have one version of (S_4) for each of these operators.

Definition 4.1 Let S be a set of judgements in $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}, \mathbb{L}^*_{\mathbb{1}}\}$. We say that a judgement γ is *provable from* (or *deducible from*) S in $\mathcal{L}$ (in symbols $S \Vdash_{\mathcal{L}} \gamma$), if there exists a sequence $\gamma_1, \ldots, \gamma_n$ of judgements ending in γ whose members are either an axiom of $\mathcal{L}$, or a member of S, or it follows from some preceding members of the sequence by using the inference rules in $\mathcal{L}$. A sequence $\gamma_1, \ldots, \gamma_n$ as above is called a *proof*.

A judgement γ is a *theorem* of $\mathcal{L}$ if it is provable in $\mathcal{L}$ from the empty set (denoted $\emptyset \Vdash_{\mathcal{L}} \gamma$, or $\Vdash_{\mathcal{L}} \gamma$).

Theorem 4.2 (Soundness of LLQ) *Let $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}, \mathbb{L}^*_{\mathbb{1}}\}$. If a judgement γ is provable from S in $\mathcal{L}$, then γ is a semantic consequence of S in $\mathcal{L}$ (in symbols, $S \Vdash_{\mathcal{L}} \gamma$ implies $S \models_{\mathcal{L}} \gamma$).*

Notation 4.3 *In LLQ we can prove that $\otimes$ is associative, commutative, and with $\top$ as null element. Thus, hereafter we will write $\phi_1 \otimes \ldots \otimes \phi_n$, or sometimes $\bigotimes_{i \leq n} \phi_i$, without involving unnecessary parenthesis, as the notion is unambiguous. This includes the case $n = 0$, where we interpret $\bigotimes_{i \leq 0} \phi_i = \top$.*

Note that, any judgement $\phi_1, \ldots, \phi_n \vdash \psi$ is provably equivalent to $\vdash (\phi_1 \otimes \ldots \otimes \phi_n) \multimap \psi$. Thus, without loss of generality, we may assume judgements are always of the form $\vdash \phi$, for some ϕ.

[5] Recall that the order on $[0, \infty]$ is reversed!

$$\frac{}{\phi \vdash \phi}\ (\text{ID}) \qquad \frac{\Gamma \vdash \phi \quad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ (\text{CUT}) \qquad \frac{\Gamma \vdash \phi}{\Gamma, \psi \vdash \phi}\ (\text{WEAK}) \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \theta}{\Gamma, \psi, \phi, \Delta \vdash \theta}\ (\text{PERM})$$

(a) Logical deduction and Structural rules

$$\frac{}{\Gamma \vdash \top}\ (\text{TOP}) \qquad \frac{}{\bot \vdash \phi}\ (\text{BOT})$$

$$\frac{\Gamma, \phi \vdash \theta}{\Gamma, \phi \wedge \psi \vdash \theta}\ (\wedge_1) \qquad \frac{\Gamma, \phi \vdash \theta \quad \Gamma, \psi \vdash \theta}{\Gamma, \phi \vee \psi \vdash \theta}\ (\vee_1)$$

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ (\wedge_2) \qquad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}\ (\vee_2)$$

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}\ (\wedge_3) \qquad \frac{\Gamma, \phi \vee \psi \vdash \theta}{\Gamma, \psi \vdash \theta}\ (\vee_3)$$

(b) Lattice rules

$$\frac{}{\vdash (\neg\phi) \vee (\neg\neg\phi)}\ (\text{WEM}) \qquad \frac{}{\vdash (\phi \multimap \psi) \vee (\psi \multimap \phi)}\ (\text{TOT})$$

$$\frac{\Gamma, \phi, \psi \vdash \theta}{\Gamma, \phi \otimes \psi \vdash \theta}\ (\otimes_1) \qquad \frac{\Gamma, \phi \multimap \theta \vdash \psi \quad \theta \vdash \phi}{\Gamma, \theta \vdash \phi \otimes \psi}\ (\multimap_1)$$

$$\frac{\Gamma, \phi \otimes \psi \vdash \theta}{\Gamma, \phi \vdash \psi \multimap \theta}\ (\otimes_2) \qquad \frac{\Gamma, \theta \vdash \phi \otimes \psi \quad \vdash \neg\neg\phi}{\Gamma, \phi \multimap \theta \vdash \psi}\ (\multimap_2)$$

$$\frac{\phi \otimes \phi \vdash \psi \otimes \psi}{\phi \vdash \psi}\ (\otimes_3) \qquad \frac{\Gamma, \theta \vdash \phi \otimes \psi \quad \vdash \neg\neg\theta}{\Gamma, \phi \multimap \theta \vdash \psi}\ (\multimap_3)$$

(c) Lawvere quantale rules

Fig. 1. The natural deduction system of $\mathbb{L}$

$$\frac{\phi \vdash \psi \quad (r > 0)}{r * \phi \vdash r * \psi}\ (S_1) \qquad \frac{}{\vdash r * (s * \phi) \multimapboth (rs) * \phi}\ (S_2)$$

$$\frac{}{\vdash \phi \multimapboth 1 * \phi}\ (S_3) \qquad \frac{\bowtie \in \{\wedge, \vee, \otimes, \multimap\}}{\vdash r * (\phi \bowtie \psi) \multimapboth r * \phi \bowtie r * \psi}\ (S_4)$$

$$\frac{}{\vdash 0 * \phi}\ (S_5) \qquad \frac{}{\vdash (r + s) * \phi \multimapboth r * \phi \otimes s * \phi}\ (S_6)$$

$$\frac{}{r * \bot \vdash \bot}\ (S_7) \qquad \frac{}{\vdash (r \dot{-} s) * \phi \multimapboth (r * \phi \multimap s * \phi)}\ (S_8)$$

$$\frac{}{\vdash r * \phi \wedge s * \phi \multimapboth \max\{r, s\} * \phi}\ (S_9)$$

$$\frac{}{\vdash r * \phi \vee s * \phi \multimapboth \min\{r, s\} * \phi}\ (S_{10})$$

Fig. 2. The natural deduction system of $\mathbb{L}^*_{\mathbb{I}}$ (scalar product rules)

4.1 Deduction Theorems

Classical and intuitionistic logics both enjoy the *deduction theorem*: if ϕ, ψ are formulas and S a set of judgements, $\vdash \psi$ is provable from $S \cup \{\vdash \phi\}$ iff $\vdash \phi \rightarrow \psi$ is provable from S. In LLQ —similarly to other substructural logics without a cancellation rule— (the left-to-right implication) does not hold.

Fact 4.4 (Failure of the deduction theorem) *Consider the formulas $\phi := \eta \wedge ((\eta \otimes \rho) \multimap \theta)$ and $\psi := \rho \multimap \theta$. Then $\vdash \psi$ is provable from $\{\vdash \phi\}$, as follows.*

$$\cfrac{\cfrac{\vdash \phi}{\vdash \eta}\,(\wedge_3) \qquad \cfrac{\vdash \phi}{\eta, \rho \vdash \theta}\,(\wedge_3\,;\otimes_2\,;\otimes_1)}{\cfrac{\rho \vdash \theta}{\vdash \psi}\,(\otimes_2)}\,(\textsc{cut})$$

But $\vdash \phi \multimap \psi$ is not provable, because otherwise, from the soundness, it should be a tautology and it is not: consider the model m such that $m(\eta) = \frac{1}{4}$, $m(\rho) = 0$ and $m(\theta) = 1$.

Similarly to other substructural logics like linear or Łukasiewicz logics, for $\mathbb{L}$ a weaker form of the deduction theorem holds: $\vdash \psi$ is provable from $S \cup \{\vdash \phi\}$ in $\mathbb{L}$ iff $\vdash n\phi \multimap \psi$ is provable from S in $\mathbb{L}$ for some $n \in \mathbb{N}$. This is a consequence of the equivalence of $\mathbb{L}$ with product logic, for which such a deduction theorem holds [13, pg.196]. However, this weaker version does not hold in $\mathbb{L}_{\mathbb{1}}$ and $\mathbb{L}^*_{\mathbb{1}}$.

Fact 4.5 (Failure of the weak deduction theorem) *Consider the formulas $\phi := \mathbb{1} \vee \neg \mathbb{1}$ and $\psi := \bot$. Then $\vdash \psi$ is provable from $\{\vdash \phi\}$ using* (ONE). *But $\vdash n\phi \multimap \psi$ is not provable for any $n \in \mathbb{N}$, since otherwise, using the soundness, there should exist an n such that $\vdash n\phi \multimap \psi$ is a tautology. However, no model satisfies this judgement. Indeed, for any model m,*

$$m(n\phi \multimap \psi) = m(\bot) \dot{-} n \min\{m(\mathbb{1}), m(\bot)\} = \infty \dot{-} n = \infty\,.$$

Fact 4.5 shows that $\mathbb{L}_{\mathbb{1}}$ is not equivalent to any of the extensions of product logic with truth-constants proposed in [4], for which such a weak deduction theorem does hold.

4.2 Totality Lemma

We call pairs of judgements of the following form

$$(\vdash \phi \multimap \psi\,,\, \vdash \psi \multimap \phi) \qquad\qquad \text{or} \qquad\qquad (\vdash \neg\phi\,,\, \vdash \neg\neg\phi)$$

supplementary judgements. These judgements, when used as a pair as above, explore different alternatives for the interpretations of LLQ-formulas. The first pair of judgements explores different ordering alternatives; the second is choosing either a finite or infinite interpretation for a formula.

Supplementary judgements play a special rôle in reasoning, as clearly stated in the following lemma.

Lemma 4.6 (Totality Lemma) *The following statements are provable in all LLQ.*

(i) *If* $\left[\dfrac{\vdash \rho \quad \phi \vdash \psi}{\vdash \theta}\ \textit{and}\ \dfrac{\vdash \rho \quad \psi \vdash \phi}{\vdash \theta}\right]$ *then* $\dfrac{\vdash \rho}{\vdash \theta}$.

(ii) *If* $\left[\dfrac{\vdash \rho \quad \vdash \neg\phi}{\vdash \theta}\ \textit{and}\ \dfrac{\vdash \rho \quad \vdash \neg\neg\phi}{\vdash \theta}\right]$ *then* $\dfrac{\vdash \rho}{\vdash \theta}$.

The totality lemma, along with others, allows us to prove the distributivity of the tensor product with respect to $\wedge$ and $\vee$, respectively.

5 Theories and Models

In what follows, we use $\mathcal{L}$ to range over $\{\mathbb{L}, \mathbb{L}_{\mathbb{1}}, \mathbb{L}_{\mathbb{1}}^*\}$, as the following definitions are uniform for all LLQ.

A *theory* $\mathbb{T}$ in $\mathcal{L}$ is a set of judgements that is deductively closed (in symbols, $\mathbb{T} \Vdash_{\mathcal{L}} \gamma$ implies $\gamma \in \mathbb{T}$). An *axiomatic theory* in $\mathcal{L}$ is a theory for which there exists a set of judgements, called *axioms*, such that all the judgements in the theory can be proven in $\mathcal{L}$ from the axioms; it is *finitely axiomatized* if it admits a finite set of axioms.

If $\mathbb{T}$ and $\mathbb{T}'$ are two theories in $\mathcal{L}$ such that $\mathbb{T} \subseteq \mathbb{T}'$, we say that $\mathbb{T}'$ is a *extension* of $\mathbb{T}$; it is a *proper extension* if $\mathbb{T} \subsetneq \mathbb{T}'$.

A theory $\mathbb{T}$ in $\mathcal{L}$ is *disjunctive*, if for any formulas $\phi, \psi \in \mathcal{L}$, $\vdash \phi \vee \psi \in \mathbb{T}$ implies that either $\vdash \phi \in \mathbb{T}$ or $\vdash \psi \in \mathbb{T}$. It is immediate that if $\mathbb{T}$ is a disjunctive theory, because of (TOT) and (WEM), we have that for any set of supplementary judgements in $\mathcal{L}$, at least one of the judgements belongs to $\mathbb{T}$.

A theory in $\mathcal{L}$ is *inconsistent* if it contains $\top \vdash \bot$, otherwise it is *consistent*; it is *maximal consistent* if it is consistent and all its proper extensions are inconsistent.

A *model of a theory* $\mathbb{T}$ is a model m that satisfies all the judgements of the theory. If the theory is axiomatized, m is a model for all the axioms iff it is a model of the theory.

Note that an assignment of values to all the propositional atoms defines a unique model since the values of all formulas are given inductively and are determined by the values of the atomic propositions.

Lemma 5.1 *In all LLQ the following statements are true.*

(i) *If a theory has a model, then it is consistent.*

(ii) *Any model satisfies a disjunctive consistent theory.*

In the case of $\mathbb{L}_{\mathbb{1}}^*$ we can identify a special class of disjunctive consistent theories.

Definition 5.2 A *diagrammatic theory* is a consistent theory $\mathbb{T}$ of $\mathbb{L}_{\mathbb{1}}^*$ such that for any $p \in \mathbb{P}$,

- either $p \vdash \bot \in \mathbb{T}$,
- or there exists $\varepsilon \in [0, \infty)$ such that $\varepsilon \vdash p \in \mathbb{T}$ and $p \vdash \varepsilon \in \mathbb{T}$.

It is not difficult to observe that in a diagrammatic theory, for any $\phi \in \mathbb{L}_{\mathbb{1}}^*$, either $\phi \vdash \bot \in \mathbb{T}$, or there exists $\varepsilon \in [0, \infty)$ such that $\varepsilon \vdash \phi,\ \phi \vdash \varepsilon \in \mathbb{T}$.

Lemma 5.3 *In $\mathbb{L}_{\mathbb{1}}^*$ we have that*

(i) *Every diagrammatic theory has a unique model.*

(ii) *Every model satisfies a unique diagrammatic theory.*

(iii) *A theory is diagrammatic iff it is maximal consistent.*

(iv) *Every disjunctive consistent theory has a unique diagrammatic extension; and a unique model.*

6 Normal Forms

In this section, we prove that any finitely axiomatized theory can be presented in a normal form, where all the axioms have a specific syntactic format.

There are some important classes of judgements that play a crucial role in our development:

$$\bot \vdash \phi \mid \phi \vdash \top \qquad \text{(tautological)}$$
$$\top \vdash \bot \mid \top \vdash \mathbb{1} \mid \mathbb{1} \vdash \bot \qquad \text{(inconsistent)}$$
$$\underbrace{\top \vdash p \mid p \vdash \bot}_{\text{alethic}} \mid \underbrace{\vdash \neg\neg p}_{\text{finitist}} \qquad \text{(assertive)}$$
$$(\textstyle\bigotimes_{i \leq n} r_i * p_i) \otimes r * \mathbb{1} \vdash (\textstyle\bigotimes_{j \leq m} s_i * q_i) \otimes s * \mathbb{1} \qquad \text{(affine)}$$

where $p, p_i, q_j \in \mathbb{P}$ are atomic propositions. In the case of $\mathbb{L}$ and $\mathbb{L}_{\mathbb{1}}$ the coefficient in an affine judgement are positive integers, and for $\mathbb{L}$ the term involving $\mathbb{1}$ is not present.

Definition 6.1 [Normal form] A judgement is in *normal form* if it is either tautological, inconsistent, assertive (finitist or alethic), or affine.

Notation 6.2 *Since in $\mathbb{L}^*_{\mathbb{1}}$ $\otimes$ commutes with all the other logical connectives, we assume hereafter that in all the formulas the scalar products guard the atomic propositions or the constants, and no other scalar products appear in a formula.*

Definition 6.3 A theory in LLQ is *normal*, or it has a *normal axiomatization*, if it admits a finite axiomatization such that

- every axiom is in normal form;
- no atomic proposition that occurs in an alethic axiom appears in any other axiom;
- there is an assertive judgement for each atomic proposition that appears in the axioms.

For LLQ it is not possible, in general, to convert a judgement into a model-theoretic equivalent judgement in normal form. It is however possible to associate with any judgement γ, a finite (possibly empty) set of normal theories $\mathbb{T}_1, \ldots, \mathbb{T}_n$, such that:

$$m \models \gamma \qquad \text{iff} \qquad \text{for some } i \leq n,\ m \models \mathbb{T}_i \ .$$

In such a case, we call the set $\{\mathbb{T}_1, \ldots, \mathbb{T}_n\}$ of theories a *normal representation of the judgement* γ. Similarly, a *normal representation of a finite set of judgements* V (or of a finitely axiomatized theory) is a finite (possibly empty) set of normal theories $\mathbb{T}_1, \ldots, \mathbb{T}_n$, such that, m is a model of V iff it is a model for at least one of the theories $\mathbb{T}_1, \ldots, \mathbb{T}_n$.

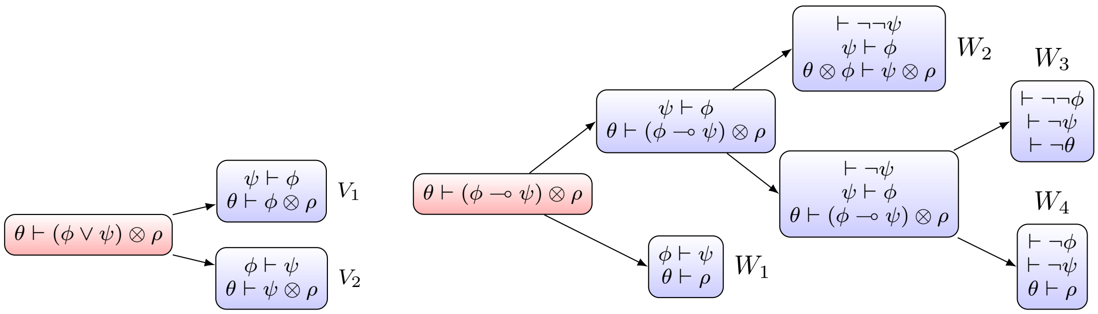
6.1 Normalization Algorithm

There exists a simple algorithm that allows us to compute, for any given finite set of judgements V, its normal representation $\mathcal{N}(V)$. The details of this algorithm are in the appendix. In what follows, we sketch how this algorithm works on a couple of examples that illustrate the main subtleties of the algorithm.

Suppose we have a finite set V of judgements. If V is not already in normal form, we can use the theorems of LLQ to simplify the judgements and eventually convert them to a normal form. In doing this, in some cases, we will have to use pairs (γ_1, γ_2) of supplementary judgements (see Section 4.2) that will be treated as new axioms. This is done when the conversion cannot progress without extra assumptions. One can think of it as "a proof by cases" resulting in two new separate set of judgements V_1 and V_2, each containing one of the supplementary judgements. The invariant preserved in each reduction step is that, for $i \in \{1, 2\}$

- all the judgements in V_i are provable from $V \cup \{\gamma_i\}$;
- all the judgements in V are provable from V_i.

Example 6.4 Let $\gamma = \theta \vdash (\phi \vee \psi) \otimes \rho$ be the judgement we would like to reduce to normal form. The disjunction occurring in γ is problematic as it prevents γ to be provably equivalent to another (single) judgement in normal form. However, by using the supplementary hypotheses $\psi \vdash \phi$ and $\phi \vdash \psi$, we can split the reduction by cases and



(a) (b)

Fig. 3. Conversion into normal representation: (a) Rule 3 and (b) Rule 5 of the normalization algorithm

obtain $V_1 = \{\phi \vdash \psi, \theta \vdash \psi \otimes \rho\}$ and $V_2 = \{\psi \vdash \phi, \theta \vdash \phi \otimes \rho\}$, two sets of judgements on which each element is (at least) one step closer to be in normal form (Fig. 3a). Note that the invariant described above is preserved.

Example 6.5 Let $\gamma = \theta \vdash (\phi \multimap \psi) \otimes \rho$ be the judgement to be converted into normal form. In this specific case, the problematic connective is $\multimap$. By adding appropriate pairs of supplementary judgements, in sequence, we split the reduction in four cases and obtain $W_1, \dots, W_4$ as new sets of judgements (Fig 3b).

Of interest in this particular case, is that in order to guarantee that the new sets of judgements have strictly reduced complexity —interpreted as number of sub-formulas not in normal form— we need to take several reduction steps.

Starting from a finite set of judgements V, the normalization algorithm works essentiality by repeatedly applying conversion rules to the judgements that are not in normal form by inspecting the structure of the formulas in the judgements. Note that, Examples 6.4 and 6.5 describe actual conversion rules in the algorithm (for the other rules see the full version of the paper on ArXiV). As each conversion rule guarantees that the number of sub-formulas not in normal form is strictly reduced, the algorithms eventually terminates.

The output $\mathcal{N}(V)$ of the algorithm is a set of theories (technically, only their axioms). The next theorem states the correctness of this conversion.

Theorem 6.6 (Normal representation) *Given a finite set V of judgements in $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}, \mathbb{L}^*_{\mathbb{1}}\}$, the set of the theories axiomatized by the elements in $\mathcal{N}(V)$ is a normal representation of the theory axiomatized by V. Consequently, any model of V is a model for at least one of the elements in $\mathcal{N}(V)$; and any model of an element in $\mathcal{N}(V)$ is a model of V.*

The normalization algorithm also allows us to prove the decidability of satisfiability in LLQ.

Theorem 6.7 (Decidability of satisfiability in LLQ) *Given a finite set V of judgements in $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}, \mathbb{L}^*_{\mathbb{1}}\}$, V is satisfiable iff there exists $S \in \mathcal{N}(V)$ s.t. $\vdash \bot \notin S$. Consequently, the satisfiability of judgements in LLQ is decidable.*

7 Completeness and Incompleteness

In this section, we demonstrate first that all the logics for the Lawvere quantale are incomplete in general, even for theories over finitely many propositional symbols. Secondly, we prove that all LLQ are complete if consider finitely-axiomatised theories only. Finally, we prove an approximate form of strong completeness over a well behaved class of theories, not necessarily finitely-axiomatizable.

Theorem 7.1 (Incompleteness) *LLQ are incomplete: for any $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}\mathbb{L}^*_{\mathbb{1}}\}$, there exist theories $\mathbb{T}$ and judgements γ in $\mathbb{L}$ so that all the models of $\mathbb{T}$ are models of γ but γ is not provable from $\mathbb{T}$ in $\mathcal{L}$. Moreover, the result is independent of the particular proof systems that are chosen for LLQ, in the sense that any finite set of finitary proof rules that can be proposed (as an alternative to the rules presented in this paper) still produces an incomplete theory for each $\mathcal{L}$.*

Proof. Consider $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}\mathbb{L}_{\mathbb{1}}^*\}$ with their proof systems presented in Section 4, or any alternative finite set of *finitary* rules that can describe $\mathcal{L}$. Let $p, q \in \mathbb{P}$ be two atomic propositions and $\mathbb{T}$ a theory in $\mathcal{L}$ axiomatized by all the judgements of the form $(n+1)p \vdash nq \quad$ for all $n \in \mathbb{N}$.

Note that in all the models m of $\mathbb{T}$ we must have $m(p) \geq m(q)$, hence all the models of $\mathbb{T}$ are also models of $p \vdash q$. Assume there exists a finite proof of $p \vdash q$ in $\mathcal{L}$ from the set $\{(n+1)p \vdash q \mid n \geq 0\}$ of axioms of $\mathbb{T}$. Since this proof is finite and uses a finite set of finitary rules, there must exist $k \geq 0$ so that the only judgements used in the proof of $p \vdash q$ are from the set $V = \{(n+1)p \vdash q \mid 0 \leq n \leq k\}$. If that is the case, then any model of V is a model for $p \vdash q$ (from soundness). But this is false: consider the model m such that $m(p) = \frac{k}{k+1}$ and $m(q) = 1$. This is a model of V, but not a model of $p \vdash q$. $\square$

A consequence of Theorem 7.1 is that not all consistent theories have models. For instance in $\mathbb{L}_{\mathbb{1}}$, the theory axiomatized by the set $\{p \vdash n \mid n \in \mathbb{N}\} \cup \{\vdash \neg\neg p\}$ of axioms for $p \in \mathbb{P}$, is consistent: any proof will use a finite subset of axioms from the first set and possibly $\vdash \neg\neg p$, and these are not sufficient to prove $\vdash \perp$. However, this theory has no model, because in any model m the axioms in the first set guarantee that $m(p) \geq n$, for all $n \in \mathbb{N}$, while the last axiom require that $m(p)$ is finite.

Theorem 7.2 (Completeness for finitely-axiomatized theories) *Let $\mathcal{L} \in \{\mathbb{L}, \mathbb{L}_{\mathbb{1}}\mathbb{L}_{\mathbb{1}}^*\}$ and $\mathbb{T}$ a finitely-axiomatized theory in $\mathcal{L}$. If a judgement γ is a semantic consequence of $\mathbb{T}$ in $\mathcal{L}$, then γ is provable from $\mathbb{T}$ in $\mathcal{L}$ (in symbols, $\mathbb{T} \models_{\mathcal{L}} \gamma$ implies $\mathbb{T} \Vdash_{\mathcal{L}} \gamma$).*

Proof. The proof is similar for all LLQ, by adapting only the arguments to the appropriate context. Hereafter, we sketch the proof for $\mathbb{L}_{\mathbb{1}}^*$, as it is the most complex of all. We only sketch this for the case both γ and $\mathbb{T}$ are in normal form. The other cases can all be reduced to this one.

Assume that γ is a normal judgement and $\mathbb{T}$ a normal theory. The cases when γ is an alethic or a finitist judgement are relatively simple. We detail here the case when γ is affine:

$$(\bigotimes_{i \leq n} r_i * p_i) \otimes r * \mathbb{1} \vdash (\bigotimes_{j \leq m} s_j * q_j) \otimes s * \mathbb{1}$$

with m, n possibly 0. We need to show that the judgement above is provable from the axioms of $\mathbb{T}$. We have a couple of cases regarding the possibility that some of atomic propositions p_i, q_j appear in alethic axioms in $\mathbb{T}$, but we can easily deal with these propositions and eventually reduce the problem to the case when none of the atomic propositions p_i, q_j appear in the alethic axioms of $\mathbb{T}$.

Consequently, all these atomic propositions appear in finitist axioms of $\mathbb{T}$. Using commutativity and associativity of $\otimes$, we will reorganise both our judgement and the non-assertive axioms of $\mathbb{T}$ so that we put together different copies of the same atomic proposition in a tensorial product and use the facts that $0p = \top$ and $r * \top \otimes \phi \dashv\vdash \phi$. So, without losing generality, we can assume that our judgement γ is

$$(\bigotimes_{i \leq k} a_i * x_i) \otimes r * \mathbb{1} \vdash (\bigotimes_{i \leq k} b_i * x_i) \otimes s * \mathbb{1}$$

and the non-assertive axioms of $\mathbb{T}$ are

$$\begin{cases} (\bigotimes_{i \leq k} a_i^1 * x_i) \otimes r^1 * \mathbb{1} \vdash (\bigotimes_{i \leq k} b_i^1 * x_i) \otimes s^1 * \mathbb{1} \\ \qquad\qquad\qquad\qquad \cdots \\ (\bigotimes_{i \leq k} a_i^l * x_i) \otimes r^l * \mathbb{1} \vdash (\bigotimes_{i \leq k} b_i^l * x_i) \otimes s^l * \mathbb{1} \end{cases}$$

for some positive reals $a_i, b_i, a_i^j, b_i^j, r, s, r^j, s^j$ and atomic propositions $x_1, \ldots, x_k$. Consider the matrices $A \in \mathbb{R}^{l \times k}$,

$C \in \mathbb{R}^{k\times 1}$ and vector $\beta \in \mathbb{R}^k$.

$$A = \begin{pmatrix} a_1^1 - b_1^1 & \dots & a_k^1 - b_k^1 \\ \dots & \dots & \dots \\ a_1^l - b_1^l & \dots & a_k^l - b_k^l \end{pmatrix} \qquad C = (b_1 - a_1, \dots, b_k - a_k) \qquad \beta = \begin{pmatrix} r^1 - s^1 \\ \dots \\ r^l - s^l \end{pmatrix}$$

and let $\delta = r - s$. According to our hypothesis, any model of $\mathbb{T}$ is a model of γ, it follows that there exists no $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ such that

$$Ax + \beta \geq 0\,, \quad Cx + \delta > 0\,.$$

By Mozkin transposition theorem [23] [6] , there exist $t_0 \in \mathbb{R}$ and $t = (t_1, \dots, t_l) \in \mathbb{R}^{l\times 1}$, such that

$$Cx + \delta = t(Ax + \beta) + t_0\,, \quad t \geq 0\,, \quad t_0 \geq 0\,.$$

In $\mathbb{L}^*_{\mathbb{1}}$ we can repeatedly apply the derived rule following the pattern from t, t_0

$$\frac{\phi_1 \vdash \psi_1 \quad \phi_2 \vdash \psi_2}{r * \phi_1 \otimes s * \phi_2 \vdash r * \psi_1 \otimes s * \psi_2}$$

we obtain a proof from $\mathbb{T}$ for $\big((\bigotimes_{i\leq k} a_i * x_i) \otimes r * \mathbb{1}\big) \vdash \big((\bigotimes_{i\leq k} b_i * x_i) \otimes s * \mathbb{1}\big)$. □

A consequence of this completeness result is the following.

Corollary 7.3 *For any finitely axiomatized theory of $\mathbb{L}^*_{\mathbb{1}}$, the set of its diagrammatic extensions coincide with the set of the diagrammatic theories of its models.*

Incompleteness over general theories (Theorem 7.1) is a common trait of several many-valued logics [21, 33, 34], especially, if interpreted over the reals. A weaker form of completeness result, first proposed by Ben Yaacov [34], is *approximate completeness*. Rather than compromise on the theories, one asks instead whether a judgement can be "proven up to arbitrary precision".

In $\mathbb{L}^*_{\mathbb{1}}$, approximate completeness can be formally stated as follows: whenever all the models of a set of judgements S are also models of $\vdash \psi$, the judgement $\epsilon \vdash \psi$ is provable from S in $\mathbb{L}^*_{\mathbb{1}}$, for any $\varepsilon > 0$.

It is not difficult to see that the above statement is still too strong to hold in $\mathbb{L}^*_{\mathbb{1}}$ for general sets of judgements S. Actually, already theories using only finitely-many atomic propositions (one is enough) can falsify the statement.

Fact 7.4 (Failure of approximate completeness) *Consider the set $S = \{p \vdash n \mid n \in \mathbb{N}\}$ of judgements in $\mathbb{L}^*_{\mathbb{1}}$, for $p \in \mathbb{P}$ a fixed atomic proposition, and take $\psi = \neg p$.*

The only model m of S is such that $m(p) = \infty$, because satisfying all the judgements of the form $p \vdash n$, for $n \in \mathbb{N}$, is equivalent to say that the interpretation of p is ∞. Thus, it is also a model for $\vdash \neg p$.

Assume that approximate completeness holds in $\mathbb{L}^*_{\mathbb{1}}$ and let $\varepsilon < \infty$. Then, $\neg p$ is provable from S. As any proof is finite, there must exists a finite subset $S' \subseteq S$ such that $\varepsilon \vdash \neg p$ is provable from S'.

Define $N := \max\{n \mid p \vdash n \in S'\}$. Then, $m'(p) := N$ is a model for S' and, by Theorem 4.2, $m \models (\varepsilon \vdash \neg p)$. That is, $\varepsilon = m'(e) \geq m'(\neg\phi)$. However, $m'(\neg\phi) = \infty$, thus, $m'(\neg\phi) > \varepsilon$ —contradiction.

Despite the fact that we cannot hope for a general form of approximate strong completeness, we can still recover a mildly restricted version of it by focusing on a suitable well-behaved class of judgements and theories over finitely-many atomic propositions.

[6] Motzkin [23] contains an unfortunate typo. For a correct form of the statement see [2]. Here we use an adapted form for affine transformations obtained by using the Fundamental Theorem of Linear Inequalities (see [27, Corollary 7.1h])

Let $\mathbb{P}_n = \{p_1, \dots, p_n\}$ be a finite set of atomic propositions, and denote by $\mathbb{L}^*_{\mathbb{1}}(n)$ the logic $\mathbb{L}^*_{\mathbb{1}}$ restricted over $\mathbb{P}_n$. Then, using the Hurwicz's general form of Farkas' Lemma [14], and following a similar proof structure as in Theorem 7.2, we can prove the following approximate completeness result.

Theorem 7.5 (Restricted approximate completeness) *Let S be a set of normal judgements in $\mathbb{L}^*_{\mathbb{1}}(n)$ such that it has only models valued over $[0, \infty)$. If a normal judgement*[7] $\vdash \psi$ *is a semantic consequence of S in $\mathbb{L}^*_{\mathbb{1}}(n)$, then for any $\varepsilon > 0$, $\varepsilon \vdash \psi$ is provable from S.*

Proof. We assume that S is not finite (the finite case is covered by Theorem 7.2) and in normal form. We further assume that the constant $\mathbb{1}$ is never used neither in S nor in ψ. This will guarantee us to work on linear maps, rather than affine ones. The generalisation to the case of affine maps can be done by invoking the Fundamental Theorem of Linear Inequalities (for details see *e.g.* [27, Corollary 7.1h]). Without loss of generality, as in Theorem 7.2, we can assume that $\vdash \psi$ is provably equivalent to

$$(\bigotimes_{i \leq n} a_i * p_i) \vdash (\bigotimes_{i \leq n} b_i * p_i)$$

and the non-assertive judgements in S are

$$\begin{cases} (\bigotimes_{i \leq n} a^1_i * p_i) \vdash (\bigotimes_{i \leq n} b^1_i * p_i) \\ (\bigotimes_{i \leq n} a^2_i * p_i) \vdash (\bigotimes_{i \leq n} b^2_i * p_i) \\ \qquad\qquad \dots \end{cases}$$

Let $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{Y} = \mathbb{R}^{\mathbb{N}}$, and denote by $\mathcal{X}^*$, $\mathcal{Y}^*$ their dual spaces, respectively (*i.e.*, $\mathcal{X}^*$ is the set of all continuous linear functionals $\mathcal{X} \to \mathbb{R}$). Define the maps $T_S \colon \mathcal{X} \to \mathcal{Y}$ and $t_\psi \colon \mathcal{X} \to \mathbb{R}$, for $x_i \in \mathbb{R}$, $j \in \mathbb{N}$ as follows

$$T_S(x_1, \dots, x_n)(j) := \sum_{i \leq n} (a^j_i - b^j_i) x_i \qquad\qquad t_\psi(x_1, \dots, x_n) := \sum_{i \leq n} (a_i - b_i) x_i$$

In abstract terms, the set of judgements in S can be thought of as the inequality $T_S(x_1, \dots, x_n) \geq 0$ and, similarly, ψ as the inequality $t_\psi(x_1, \dots, x_n) \geq 0$. Define the sets

$$\begin{aligned} V_S &= \{x^* \in \mathcal{X}^* \mid \forall x \in \mathcal{X}.\, T_S(x) \geq 0 \text{ implies } x^*(x) \geq 0\}\,, \\ Z_S &= \{x^* \in \mathcal{X}^* \mid \exists y^* \in \mathcal{Y}^*.\, x^* = T^*_S(y^*) \text{ and } y^* \geq 0\}\,. \end{aligned}$$

where $T^*_S \colon \mathcal{Y}^* \to \mathcal{X}^*$ is the adjoint of T_S uniquely defined by the adjoint property as $T^*_S(y^*) := y^* \circ T_S$.

By Hurwicz's general form of Farkas' Lemma [14], we know that V_S is the *regularly convex envelope* of Z_S, which corresponds to the topological closure $\overline{Z_S}$ of Z_S for finite-dimensional vector spaces as $\mathcal{X}^*$ is. Let $\pi_k \colon \mathcal{Y} \to \mathbb{R}$ denote the k^{th}-projection function defined as $\pi_k((x_i)_{i \in \mathbb{N}}) = x_k$. Clearly $\pi_k \in \mathcal{Y}^*$. Moreover, the finite positive linear combinations of these projections, forms a dense subset P of $\mathcal{Y}^*_+ = \{y^* \mid y^* \geq 0\}$. Notice that $Z_S = T^*_S(\mathcal{Y}^*_+)$. Since T^*_S is a continuous function, $\overline{Z_S} = \overline{T^*_S(P)}$. As we have established that $V_S = \overline{Z_S}$, any element of V_S can be approached arbitrary close by a map of the form $T^*_S(p)$, for $p \in P$. In simpler terms, as $T^*_S(p) = p \circ T_S$, any element of V_S is arbitrary close to a finite positive linear combination of judgements in S.

By hypothesis, ψ is a semantic consequence of S, meaning that, $t_\psi \in V_S$. From this, to prove $\vdash \psi$ from S in $\mathbb{L}^*_{\mathbb{1}}$ we just need to pick an appropriate $p \in P$ (which exists) and replicate the finite positive linear combination it represents by using the derived rule

$$\frac{\phi_1 \vdash \psi_1 \quad \phi_2 \vdash \psi_2}{r * \phi_1 \otimes s * \phi_2 \vdash r * \psi_1 \otimes s * \psi_2}$$

[7] By an abuse of notation, here we actually mean that $\vdash \phi$ is provably equivalent to a judgement in normal form.

to obtain a judgement $\vdash \psi'$ that is ε-close to $\vdash \psi$. As we assumed ψ to be provably equivalent to $\vdash \psi$, by chaining the two proofs together, we are done. □

8 Inference Systems for the Lawvere Quantale

In this section, we extend further the concept of proof systems in LLQ and discuss *inference systems*. These are obtained by requiring that a theory in $\mathbb{L}^*_{\mathbb{1}}$ obeys extra inferences (proof rules), in addition to its axioms and to the inferences of $\mathbb{L}^*_{\mathbb{1}}$. Because in LLQ inferences cannot be internalized as judgements (see Fact 4.5), the effect of closing a theory by a rule does not always produce another theory, as happens in classical logics, but often a set of theories.

Note that, all the inferences of LLQ have finite sets of hypotheses and so, when one works with theories of LLQ, will only have derived rules that contains a finite set of hypotheses. But if we now allow ourselves to work with inferences that might not be derived from the proof rules, we might have to handle inferences with a countable set of hypotheses. In fact, for our purpose, we are interested in only one type of such inferences that we shall call *inductive inferences*.

Definition 8.1 [Inductive inferences] An *inductive inference* in LLQ is an inference of type

$$\frac{\{\vdash \phi_i \mid i \in \mathbb{N}\}}{\vdash \psi}$$

such that for any $i, j \in \mathbb{N}$ with $i < j$, $\phi_j \vdash \phi_i$.

Observe that the inferences with a finite number of hypotheses are all particular cases of inductive inferences, since they all can be equivalently represented firstly as inferences with only one hypothesis (the conjunction of all the hypotheses); and secondly, this hypothesis can be seen as a constant sequence of hypotheses. Hence, all the proof rules of LLQ and all the inferences that can be derived from them are inductive inferences. Last but not least, observe that any axiom $\vdash \psi$ can be seen as an inductive inference with an empty set of hypotheses.

Definition 8.2 A theory $\mathbb{T}$ of $\mathbb{L}^*_{\mathbb{1}}$ is *closed under the inductive inference*

$$\frac{\{\vdash \phi_i \mid i \in \mathbb{N}\}}{\vdash \psi}\ (\mathrm{I})$$

if for any diagrammatic extension $\mathbb{T}^+$ of $\mathbb{T}$ we have that

- either $\vdash \psi \in \mathbb{T}^+$,
- or there exists $\varepsilon \in [0, \infty)$, $\varepsilon > 0$ and $i \in \mathbb{N}$ such that $\phi_i \vdash \varepsilon \in \mathbb{T}^+$.

When a model m is such that $\{\vdash \phi_i \mid i \in I\} \models_m \vdash \psi$, we say that it is a model of the inference I.

Observe that the previous definition makes sense semantically, since it implies that a theory $\mathbb{T}$ of $\mathbb{L}^*_{\mathbb{1}}$ is closed under the inference I iff any model m of $\mathbb{T}$ is a model of I.

Definition 8.3 [Inference System] An *inference system in* $\mathbb{L}^*_{\mathbb{1}}$ is a set $\mathcal{R}$ of inductive inferences in $\mathbb{L}^*_{\mathbb{1}}$.

We say that a model satisfies an inference system if it is a model for each inference in the system.

A theory is closed with respect to an inference system, if it is closed under each inference in the system.

Since we can see any axiom of a theory as a particular inference with an empty set of hypotheses, we see that any finite axiomatic system of LLQ is, in fact, a particular case of an inference system. There exists, however, interesting mathematical theories, such as the quantitative equational logic, that cannot be presented by using an axiomatic system in LLQ, but only by using an inference system.

Because the finite axiomatic systems in LLQ are particular type of inference systems, we can read the completeness theorem for finitely-axiomatized theories proven before, as a particular case of completeness for inference systems. In what follows, we will enforce these results and prove completeness results directly for inference systems.

Theorem 8.4 (Completeness for inference systems) *Let $\mathcal{R}$ be an inference system of $\mathbb{L}^*_{\mathbb{1}}$ and*

$$\frac{\{\vdash \phi_i \mid i \in \mathbb{N}\}}{\vdash \psi}\ (\mathrm{I})$$

an inductive inference. If I is satisfied by all the models of $\mathcal{R}$, then all the finitely axiomatized theories closed under $\mathcal{R}$ are also closed under I.

Proof. Let $\mathbb{T}$ be a finitely axiomatized theory closed under $\mathcal{R}$. Then any model m of $\mathbb{T}$ satisfies I, hence

- either $m \models (\vdash \psi)$
- or there exist $i \in \mathbb{N}$ and $\varepsilon > 0$ such that $m \models (\phi_i \vdash \varepsilon)$.

Applying Corollary 7.3, to m corresponds a diagrammatic theory $\mathbb{T}_m$ such that $m \models (\phi \vdash \phi')$ implies $\phi \vdash \phi' \in \mathbb{T}_m$; and to each diagrammatic extension $\mathbb{T}^+$ of $\mathbb{T}$ corresponds a model $m_{\mathbb{T}^+}$ such that $m_{\mathbb{T}^+} \models (\phi \vdash \phi')$ implies $\phi \vdash \phi' \in \mathbb{T}^+$.

Consider now an arbitrary diagrammatic extension $\mathbb{T}^+$ of $\mathbb{T}$. Since $m_{\mathbb{T}^+}$ is a model of $\mathbb{T}$, we have that

- either $m_{\mathbb{T}^+} \models (\vdash \psi)$, implying $\vdash \psi \in \mathbb{T}^+$,
- or there exist $i \in \mathbb{N}$ and $\varepsilon > 0$ such that $m_{\mathbb{T}^+} \models (\phi_i \vdash \varepsilon)$, implying $\phi_i \vdash \varepsilon \in \mathbb{T}^+$.

Hence, $\mathbb{T}$ is closed under I. $\square$

8.1 *The inference system of Quantitative Algebra*

In this section, we show how we can use LLQ as a support for quantitative equational reasoning [20]. Quantitative algebras [20] have been introduced, as a generalization of universal algebras, meant to axiomatize not only congruences, but algebraic structures over extended metric spaces. Given an algebraic similarity type Ω, a quantitative algebra is an Ω-algebra supported by an extended metric space, so that all the algebraic operators are nonexpansive. Such a structure can be axiomatized using an extension of equational logics that uses, instead of equations of type $s = t$ for some terms s, t, quantitative equations of type $s =_\varepsilon t$ for some $\varepsilon \in [0, \infty)$. This quantitative equation is interpreted as "the distance between the interpretation of s and t is less or equal to ε".

In the theory of quantitative algebras, $=_\varepsilon$ are treated as classic Boolean predicates, so in any model, $s =_\varepsilon t$ is either *true* or *false*. However, a different way to look to this, is to actually think that we only have one equality predicate and $s = t$ is interpreted in the Lawvere quantale, thus allowing us to reason about the distance between s and t. For instance, instead of $s =_\varepsilon t$, we could could use the syntax of $\mathbb{L}^*_{\mathbb{1}}$, treate $s = t$ as an atomic proposition, and write $\varepsilon \vdash s = t$. This allows us to properly reason about extended metric spaces and encode, in our logic, the entire theory of quantitative equational reasoning.

In this section, we show how such an encoding is defined. $\mathbb{L}^*_{\mathbb{1}}$ has already all the necessary ingredients to do this work. However, since $\mathbb{L}^*_{\mathbb{1}}$ is only propositional, the way to do this is to treat all the equations as atomic propositions. This is exactly how we encode the classic equational logic into Boolean propositional logic. And, as in the classic case, while this is sufficient, it unfortunately requires an infinite set of axioms.

As we have already anticipated, the theory of quantitative equational logic requires an inference system in $\mathbb{L}^*_{\mathbb{1}}$, and it cannot be only encoded using an axiomatic system. This is because a judgement of type $s =_\varepsilon t \vdash s' =_\delta t'$ in quantitative reasoning corresponds to the inference $\dfrac{\varepsilon \vdash s = t}{\delta \vdash s' = t'}$ which cannot be internalized due to the fact that in $\mathbb{L}^*_{\mathbb{1}}$ the deduction theorem fails.

Concretely, assuming an algebraic similarity type Ω and a set X of variables, we construct all the possible algebraic terms. Let ΩX be the set of these terms. We define $\mathbb{L}^*_{\mathbb{1}}$ for $\mathbb{P} = \{s = t \mid s, t \in \Omega X\}$. This gives us the syntax we need.

The original axioms of quantitative equational logic are presented in Table 1, where they are stated for arbitrary terms $s, t, u, s_1, \ldots, s_n, t_1, \ldots, t_n \in \Omega X$, for arbitrary n-ary operator $f : n \in \Omega$, arbitrary positive reals $\varepsilon, \varepsilon' \in [0, \infty)$

(Refl)	$\vdash t =_0 t$
(Symm)	$s =_\varepsilon t \vdash t =_\varepsilon s$
(Triang)	$t =_\varepsilon u, u =_{\varepsilon'} s \vdash t =_{\varepsilon+\varepsilon'} s$
(Max)	$s =_\varepsilon t \vdash s =_{\varepsilon+\varepsilon'} t$
(Nexp)	$\{s_i =_\varepsilon t_i \mid i \leq n\} \vdash f(s_1, \ldots, s_n) =_\varepsilon f(t_1, \ldots, t_n)$
(Cont)	$\{s =_{\varepsilon_i} t \mid i \in \mathbb{N}\} \vdash s =_\varepsilon t$

Table 1
Quantitative algebras

$$\frac{}{\vdash t = t}\ (\text{REFL}) \qquad \frac{\varepsilon \vdash s = t}{\varepsilon \vdash t = s}\ (\text{SYMM}) \qquad \frac{\varepsilon \vdash t = u \quad \varepsilon' \vdash u = s}{\varepsilon \otimes \varepsilon' \vdash t = s}\ (\text{TRIANG}) \qquad \frac{\varepsilon \vdash s = t}{\varepsilon \otimes \varepsilon' \vdash s = t}\ (\text{MAX})$$

$$\frac{\varepsilon \vdash s_1 = t_1 \quad \ldots \quad \varepsilon \vdash s_n = t_n}{\varepsilon \vdash f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n)}\ (\text{NEXP}) \qquad \frac{\varepsilon_1 \vdash s = t \quad \ldots \quad \varepsilon_i \vdash s = t \quad \ldots}{\varepsilon \vdash s = t}\ (\text{CONT})$$

Table 2
Quantitative algebras

and arbitrary decreasing convergent sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of positive reals with limit ε. These axioms, together with the standard substitution, cut and assumption rules, provide the proof system of quantitative equational logics.

When translated into LLQ, the substitution, cut, and assumption rules are embedded in the way a proof operates. And the axioms of quantitative equational logic can be translated into the corresponding inferences in Table 2.

However, in this translation, the set of axioms is actually infinite, because the terms equalities are names for atomic propositions and, as such, we will have, for instance, a (REFL) inference rule for each term t, a (SYMM) inference rule for each tuple of terms s and t, etc. This is not surprising, as the same situation happens when we encode the classic equational logic developed for universal algebras in propositional logic.

Observe also that the axiom (CONT) of quantitative equational logic, which is an infinitery axiom, is translated into $\mathbb{L}^*_{\mathbb{1}}$ as an inductive inference rule. Indeed, first of all, we can convert each hypothesis of type $\varepsilon_i \vdash s = t$ into the equivalent one, $\vdash \varepsilon_i \multimap (s = t)$. And secondly, because for $i \geq j$ we have $\varepsilon_i \leq \varepsilon_j$, and this implies that $\varepsilon_j \multimap (s = t) \vdash \varepsilon_i \multimap (s = t)$ is a theorem in $\mathbb{L}^*_{\mathbb{1}}$.

A limitation of this encoding comes from the fact that we need an infinite set of inductive inferences to rule the quantitative equational reasoning. But this is similar with what is happening in the classical logic, when one encodes the classic equational logic into propositional logic. And as in the classic case, this can be avoided by extending $\mathbb{L}^*_{\mathbb{1}}$ with predicates. This would allow us to present a more compact and finitary inference system. We leave this extension for future work.

9 Conclusions

In this paper, we developed three propositional logics interpreted in the Lawvere quantale. We develop natural deduction systems for them, which collect rules similar to rules well-known from other logics. We show that despite their natural arithmetic interpretation, these logics manifest important metatheoretical original features that differentiate them from other related logics. We prove that the logics are incomplete in general, but complete if we restrict to finitely-axiomatized theories. We present a normalization algorithm that proves the consequence is decidable and, when made efficient, establishes complexity bounds. Although one of our logics and results for it are known in the context of product logics, our proofs are novel in all cases.

We also show that quantitative equational logic can all be encoded in our new settings. Moreover, we demonstrate that for this class of logics one can either use axiomatic systems or systems of inferences, the second providing a higher expressivity from the point of view of mathematical theories that can be developed.

It would be interesting to extend our logic to allow products as well as sums. In that case we would be interested in systems of polynomial inequalities, when the Krivine–Stengle Positivstellensatz would surely come into play [17, 29] in place of the Farkas' Lemma.

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