Final coalgebras in categories with factorization systems

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Abstract. For a functor $T: \mathcal{C} \to \mathcal{C}$, we show that, if $\mathcal{C}$ admits a factorization system $(\mathcal{L}, \mathcal{R})$ with all arrows in $\mathcal{R}$ monic and $\mathcal{C}$ is $\mathcal{R}$-well-powered, then the final coalgebra is characterized constructively as the $\mathcal{R}$-union of certain sets of T-coalgebras, provided that the final sequence of T has an $\mathcal{R}$-arrow at some ordinal α, and T preserves $\mathcal{R}$-morphisms.

Final sequence: Let T be an endofunctor on a category $\mathcal{C}$ with final object and limits of ordinal-indexed diagrams. The *final sequence* of T is a limit-preserving functor $A: \mathbf{Ord}^{op} \to \mathcal{C}$ such that, for all ordinals $\gamma \leq \beta$, $A(\beta{+}1) = TA(\beta)$, $A(\beta{+}1 \to \gamma{+}1) = TA(\beta \to \gamma)$, and $A(0) = 1$. Note that, since A preserves limits, for all limit ordinals β, the arrow $A(\beta) \to \lim_{\gamma<\beta} A(\gamma)$ is an isomorphism.

In [1,2], it is shown that if this sequence *stabilizes* at some α, in the sense that $f = A(\alpha{+}1 \to \alpha)$ is an isomorphism, then $(A(\alpha), f^{-1})$ is a final T-coalgebra. This follows since, for any T-coalgebra (X, h) and ordinal β, there exists a cone $(X, (h_\gamma)_{\gamma \in \beta^{op}})$ on $A{\restriction}\beta$[1] uniquely determined by $A(\gamma{+}1 \to \gamma) \circ Th_\gamma \circ h = h_\gamma$, for all $\gamma \leq \beta$. Therefore, $h_\alpha : (X, h) \to (A(\alpha), f^{-1})$ is a morphism of T-coalgebras, which is easily seen to be unique. Note that, this method is constructive if one can determine an ordinal α at which the final sequence stabilizes.

In [3], Worrell shows that for any mono-preserving accessible endofunctor on a locally presentable category the final sequence stabilizes. However, the proof does not give any constructive bound for stabilization. If one restricts the attention to only κ-accessible endofunctors on **Set**, $\kappa + \kappa$ steps are sufficient for the final sequence to stabilize. This bound depends heavily on the fact that in **Set**, all monomorphisms split, which is indeed a very strong requirement.

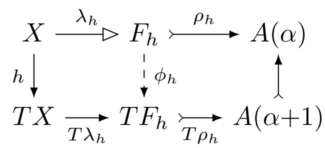
Both the results need that the final sequence A reaches a monic arrow at some α, then stabilization follows since the category is well-powered and all $A(\gamma)$ are subobjects of $A(\alpha)$, for all $\gamma \geq \alpha$. The restriction on accessible functors on locally accessible categories ensures that these requirements hold, and that the underlying category has a (strong-epi, mono) factorization system.

A new characterization: Let $(\mathcal{L}, \mathcal{R})$ be a factorization system for $\mathcal{C}$, such that arrows in $\mathcal{R}$ are monic, and let $\mathcal{C}$ be $\mathcal{R}$-well-powered. Under these hypotheses, we give a characterization of a final coalgebra for a functor $T: \mathcal{C} \to \mathcal{C}$ via its final sequence. The use of the final sequence is twofold: it guarantees unicity of

[1] $A{\restriction}\beta : \beta^{op} \to \mathcal{C}$ restricts A on the full subcategory of $\mathbf{Ord}^{op}$ of all ordinals $\gamma \leq \beta$.

the final homomorphism, and provides a weakly final coalgebra. Notably, we do not need any bound for stabilization, still the proof is constructive.

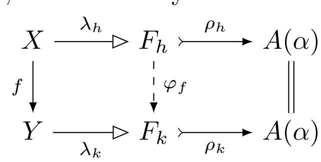
Theorem 1. *Assume A, the final sequence of T, is such that $A(\alpha{+}1 \to \alpha) \in \mathcal{R}$, for some α, and that T preserves $\mathcal{R}$-morphisms. Then, for any T-coalgebra (X, h) and $\rho_h \circ \lambda_h$ $(\mathcal{L}, \mathcal{R})$-factorization of h_α, there exists a unique arrow ϕ_h making the diagram aside commute.*



Moreover, if (X, h) is weakly final, then (F_h, ϕ_h) is a final T-coalgebra.

Proof. (Sketch) Since T preserves $\mathcal{R}$-morphisms, $T\rho_h \in \mathcal{R}$. Therefore, the outer square diagram is a lifting problem for $\lambda_h \in \mathcal{L}$ and $A(\alpha{+}1 \to \alpha) \circ T\rho_h \in \mathcal{R}$, and ϕ_h is solution to it. Assume (X, h) is weakly final, then (F_h, ϕ_h) is weakly final too, since λ_h is a T-homomorphism. Unicity follows by left cancellability of ρ_h, since, for any T-coalgebra (Y, k) and arrow $f : Y \to A(\alpha)$ such that $f = A(\alpha{+}1 \to \alpha) \circ Tf \circ k$, one proves by transfinite induction that $f = k_\alpha$. □

Note that, $(\mathcal{L}, \mathcal{R})$-factorizations of morphisms are not unique, hence for a given T-coalgebra (X, h), the associated T-coalgebra (F_h, ϕ_h) is not uniquely determined. However, under the hypothesis of Theorem 1, one can fix any factorization $h_\alpha = \rho_h \circ \lambda_h$ to obtain an endofunctor F on the category of T-coalgebras, mapping objects (X, h) to (F_h, ϕ_h), and morphisms $f : (X, h) \to (Y, k)$ to the unique solution φ_f of the lifting problem on the right. Functoriality crucially depends on the assumption that all $\mathcal{R}$-morphisms are monic (see the Appendix for details).



Finally, observe that, for all T-coalgebras (X, h), F_h is an $\mathcal{R}$-subobject of $A(\alpha)$, and since $\mathcal{C}$ is assumed to be $\mathcal{R}$-well-powered, there must be only a set I (up to isomorphism) of such F_h's. Thus, if $\mathcal{C}$ has coproducts, we are allowed to take the coproduct coalgebra $\coprod_I (F_i, \phi_i)$, which is readily seen to be weakly final, with homomorphism from any T-coalgebra (X, h) given by

$$X \xrightarrow{\lambda_h} F_h \xrightarrow{\cong} F_i \xrightarrow{in_i} \coprod F_i$$

where F_i is the representative of F_h in I.

Applying Theorem 1 to $\coprod_I (F_i, \phi_i)$, the final T-coalgebra is just the $\mathcal{R}$-union of the coalgebras in I. Notably, finality does not depend on the choice of I, which can be determined constructively by an analysis on the $\mathcal{R}$-subobjects of $A(\alpha)$.

References

1. J. Adámek and V. Koubek. On the greatest fixed point of a set functor. *Theoretical Comput. Sci.*, 150(1):57–75, 1995.
2. Michael and Barr. Algebraically compact functors. *Journal of Pure and Applied Algebra*, 82(3):211 – 231, 1992.
3. J. Worrell. Terminal sequences for accessible endofunctors. *Electronic Notes in Theoretical Computer Science*, 19:24–38, 1999.

Appendix

Here we provide full proofs of all the technical statements mentioned above.

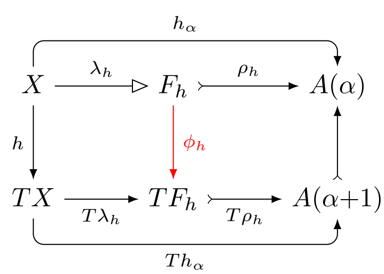
Lemma 2. *Let A be the final sequence of T. For any T-coalgebra (X,h) and ordinal α, if a morphism $k\colon X \to A(\alpha)$ is such that $k = A(\alpha{+}1 \to \alpha) \circ Tk \circ h$, then $k = h_\alpha$.*

Proof. For $\beta \leq \alpha$, let $k_\beta = A(\alpha \to \beta) \circ k$. We will show by transfinite induction that $k_\beta = h_\beta$, for $\beta \leq \alpha$. Certainly $k_0 = h_0$ since their codomain is the final object. Assuming $k_\beta = h_\beta$, we have

$$\begin{aligned}
k_{\beta+1} &= A(\alpha \to \beta{+}1) \circ k && \text{(by def. } k_{\beta+1}\text{)}\\
&= A(\alpha \to \beta{+}1) \circ A(\alpha{+}1 \to \alpha) \circ Tk \circ h && \text{(by hp.)}\\
&= A(\alpha{+}1 \to \beta{+}1) \circ Tk \circ h && \text{(by funct. } A\text{)}\\
&= TA(\alpha \to \beta) \circ Tk \circ h && \text{(by def. } A\text{)}\\
&= Tk_\beta \circ h && \text{(by def. of } k_\beta\text{)}\\
&= Th_\beta \circ h && \text{(by inductive hp.)}\\
&= h_{\beta+1} && \text{(by def. } h_{\beta+1}\text{)}
\end{aligned}$$

Suppose β is a limit ordinal and $k_\gamma = h_\gamma$, for every $\gamma < \beta$. By definition of k_γ and by the fact that $(X, (h_\gamma)_{\gamma \in \alpha^{op}})$ is a cone on $A{\restriction}\alpha$, $A(\alpha \to \gamma) \circ k = A(\alpha \to \gamma) \circ h_\alpha$. Since $A(\beta) \cong \lim_{\gamma<\beta} A(\gamma)$, we have $A(\alpha \to \beta) \circ k = A(\alpha \to \beta) \circ h_\alpha$. Therefore, by definition of k_β and compatibility of the cone, $k_\beta = h_\beta$. □

Proof. (Theorem 1) By hypothesis T preserves $\mathcal{R}$-morphisms, hence $T\rho_h \in \mathcal{R}$. Therefore $A(\alpha{+}1 \to \alpha) \circ T\rho_h \in \mathcal{R}$, since by hypothesis $A(\alpha{+}1 \to \alpha) \in \mathcal{R}$ and $\mathcal{R}$ is closed by composition. By definition, $h_\alpha = A(\alpha{+}1 \to \alpha) \circ Th_\alpha \circ h$, therefore, the diagram below is a lifting problem for $\lambda_h \in \mathcal{L}$ and $A(\alpha{+}1 \to \alpha) \circ T\rho_h \in \mathcal{R}$



hence, ϕ_h can be chosen as a solution to it, so that all the sub-diagrams commute.

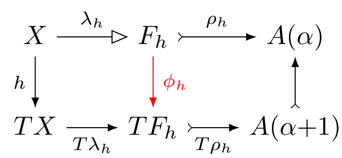
Assume (X,h) is weakly final. We will prove that (F_h, ϕ_h) is a final T-coalgebra. Let (Y,k) be a T-coalgebra, then, by weakly finality of (X,h), there exists a morphism $f\colon (Y,k) \to (X,h)$. Note that, $\lambda_h\colon (X,h) \to (F_h,\phi_h)$ is a morphism of T-coalgebras, therefore $\lambda_h \circ f$ is a T-homomorphism from (Y,k) to (F_h,ϕ_h). This proves weak finalilty. As for unicity, let $f,g\colon (Y,h) \to (F_h,\phi_h)$ be two morphisms of T-coalgebras. Consider the two composites $\lambda_h \circ f$ and $\lambda_h \circ g$. Since both composites satisfy the conditions of Lemma 2, we have that

$\lambda_h \circ f = \lambda_h \circ g$. Now, sice $\lambda_h \in \mathcal{R}$ and all $\mathcal{R}$-morphisms are monic, by left cancellabilty of monomorphisms we conclude that $f = g$. □

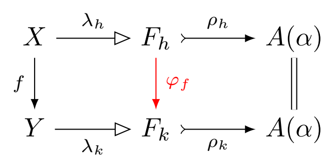
Another interesting consequence of the fact that all arrows in $\mathcal{R}$ are monic is that the mapping $(X, h) \mapsto (F_h, \phi_h)$ in Theorem 1, can be extended to a functor.

Proposition 3. *Let A be the final sequence of T. Assume that T preserves $\mathcal{R}$-morphisms and that $A(\alpha{+}1 \to \alpha) \in \mathcal{R}$, for some α. Then, we can define a functor $F\colon T\text{-}\mathsf{Coalg} \to T\text{-}\mathsf{Coalg}$ as follows:*

Objects: *For a T-coalgebra (X, h), we define $F(X, h) = (F_h, \phi_h)$, where $\rho_h \circ \lambda_h$ is a distinguished $(\mathcal{L}, \mathcal{R})$-factorization of h_α, and ϕ_h is the (unique) solution to the lifting problem below.*



Morphisms *For a morphism $f\colon (X, h) \to (Y, k)$ of T-coalgebras, we define $Ff = \varphi_f$, where φ_f is the (unique) solution to the lifting problem below.*



Proof. We first check that F is well defined. Let $f\colon (X, h) \to (Y, k)$ be a morphism of T-coalgebras, then we have to prove that $Ff = \varphi_f\colon F_h \to F_k$ is a homomorphism of T-coalgebras. By definition, we have

$$\begin{aligned} A(\alpha{+}1 \to \alpha) \circ T\rho_k \circ T\varphi_f \circ \phi_h &= A(\alpha{+}1 \to \alpha) \circ T\rho_h \circ \phi_h && \text{(by def. } T\varphi_f\text{)} \\ &= \rho_h && \text{(by def. } \phi_h\text{)} \\ &= \rho_k \circ \varphi_f && \text{(by def. } \varphi_f\text{)} \\ &= A(\alpha{+}1 \to \alpha) \circ T\rho_k \circ \phi_k \circ \varphi_f && \text{(by def. } \rho_k\text{)} \end{aligned}$$

Since T preserves $\mathcal{R}$-morphisms and $\mathcal{R}$ is closed under composition, we have that $A(\alpha{+}1 \to \alpha) \circ T\rho_k \in \mathcal{R}$, therefore is monic. By left cancellability of monomorphisms, we have $T\varphi_f \circ \phi_h = \phi_k \circ \varphi_f$, hence φ_f is a T-homomorphism.

Fuctoriality is easily proved. Let $f\colon (X, h) \to (Y, k)$ and $g\colon (Y, k) \to (Z, l)$ be morphisms in T-Coalg. By definition $\rho_h = \rho_l \circ F(g \circ f)$, $\rho_h = \rho_k \circ F(f)$, and $\rho_k = \rho_l \circ F(g)$, therefore $\rho_l \circ F(g \circ f) = \rho_l \circ F(g) \circ F(f)$. By left cancellability of ρ_l we have $F(g \circ f) = F(g) \circ F(f)$. Consider $Fid_{(X,h)}$. By definition, we have $\rho_h \circ Fid_{(X,h)} = \rho_h$, therefore $\rho_h \circ Fid_{(X,h)} = \rho_h \circ id_X$. Again, by left cancellability of ρ_h, we have $Fid_{(X,h)} = id_X$, and since $id_X = id_{(X,h)}$, we have $Fid_{(X,h)} = id_{(X,h)}$. □