Complete Axiomatization for the Bisimilarity Distance on MCs

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Introduction

- **Kleene's Theorem**: fundamental correspondence between regular expressions and DFAs
- **Salomaa'66**, **Kozen'91**: complete axiomatization for proving equivalence of regular expressions
- **Milner'84**: applied the above program on process behaviors and LTSs
- Many variations of the above schema

Example: Markov chains

Expressions: $t, s := X \mid \mathbf{a}.t \mid t +_e s \mid \mathrm{rec}\ X.t$

Example: Markov chains

names
$X \in \mathbb{X}$

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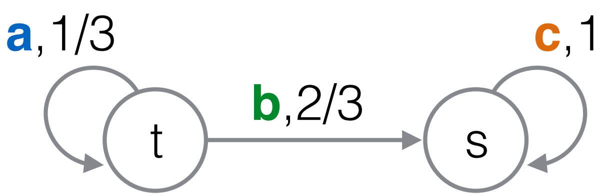
Kleene's theorem for MCs



⟺

$t = \text{rec}\ X.(a.X +_{1/3} b.s)$

$s = \text{rec}\ Y.(c.Y)$

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finite MCs

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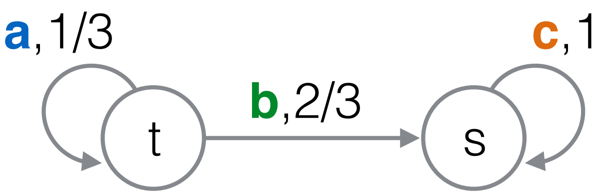

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Milner's recursion axioms

(Unfold) $\vdash$ rec X.t = t[rec X.t / X]

(Fix) {t = s[t / X]} $\vdash$ t = rec X.s — for X guarded in t

(Unguard) $\vdash$ rec X.(t $+_e$ X) = rec X.t

…for probabilistic systems

- **Generative Markov chains:**
  Baeten-Bergstra-Smolka'95 & Stark-Smolka'00
- **Simple Probabilistic Automata:**
  Bandini-Segala'01
- **(fully) Probabilistic Automata:**
  Mislove-Ouaknine-Worrell'04 (strong-bisimulation)
  Deng-Palamidessi'07 (weak-bisimulation & behavioral eq.)
- **Quantitative Kleene Coalgebras:**
  Silva-Bonchi-Bonsangue-Rutten'11 (coagebraic bisim.)

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$$s = t \quad \Longrightarrow \quad s =_{\varepsilon} t$$

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completeness almost for free!

$$s =_{\varepsilon} t$$

Equational Theories

$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

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$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

(Refl) $\vdash t = t$

(Symm) $\{t = s\} \vdash s = t$

(Trans) $\{t = u, u = s\} \vdash t = s$

(Cong) $\{t_1 = s_1, \ldots, t_n = s_n\} \vdash f(t_1, \ldots, t_n) = f(s_1, \ldots s_n)$ — for $f \in \Sigma$

Quantitative Theories

Mardare-Panangaden-Plotkin (LICS'16)

$$\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_\varepsilon s$$

quantitative inference

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

(NExp) $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1, \ldots, t_n) =_\varepsilon f(s_1, \ldots s_n)$ — for $f \in \Sigma$

(Max) $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta > 0$

(Arch) $\{t =_\delta s \mid \delta > \varepsilon\} \vdash t =_\varepsilon s$

Quantitative Semantics

Quantitative Algebra

$\mathscr{A} = (A, \Sigma_A, d_A)$ → (A, Σ_A) — Universal algebra; (A, d_A) — metric space

Satisfiability

$$\mathscr{A} \models \left(\{ t_i =_{\varepsilon_i} s_i \mid i \in I \} \vdash t =_{\varepsilon} s \right)$$

iff

for all $i \in I$. $d_A([\![t_i]\!], [\![s_i]\!]) \leq \varepsilon_i$ implies $d_A([\![t]\!], [\![s]\!]) \leq \varepsilon$

completeness

quantitative algebra

quantitative theory

$$\mathscr{A} \models \left(\vdash t =_{\varepsilon} s \right) \qquad \left(\vdash t =_{\varepsilon} s \right) \in \mathscr{U}$$

soundness

completeness

quantitative algebra

quantitative theory

$$\mathscr{A}_{MC} \models \left(\vdash t =_\varepsilon s \right) \qquad \left(\vdash t =_\varepsilon s \right) \in \mathscr{U}_{MC}$$

soundness

The Quantitative Universal Algebra

Universal Algebra of MCs

Signature: $X : 0 \mid a.- : 1 \mid +_e : 2 \mid \text{rec}\, X : 1$

Universal Algebra of MCs

Signature: X : 0 | a.- : 1 | $+_e$: 2 | rec X : 1

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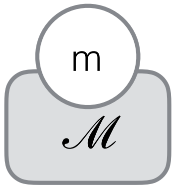
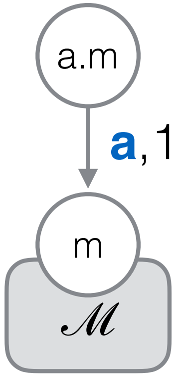
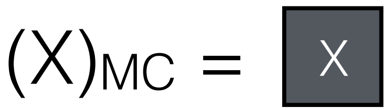
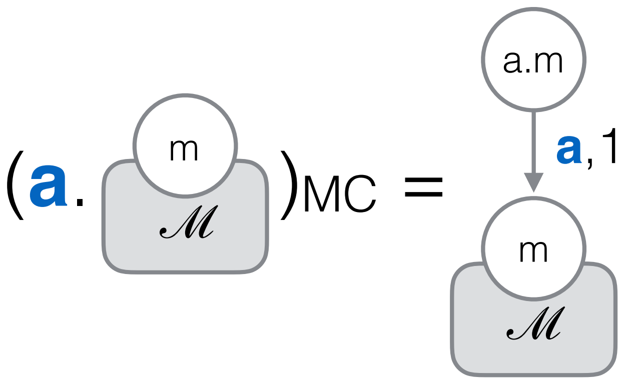
(**a**. 

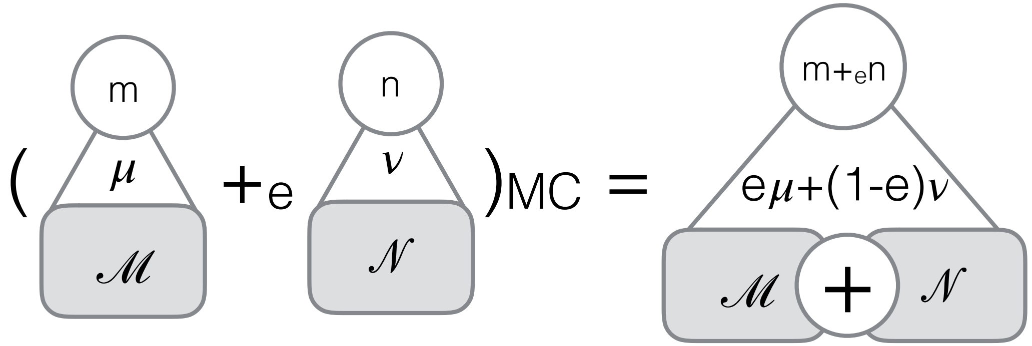
$)_{MC}$ = 

Universal Algebra of MCs

Signature: X : 0 | a.- : 1 | $+_e$: 2 | rec X : 1







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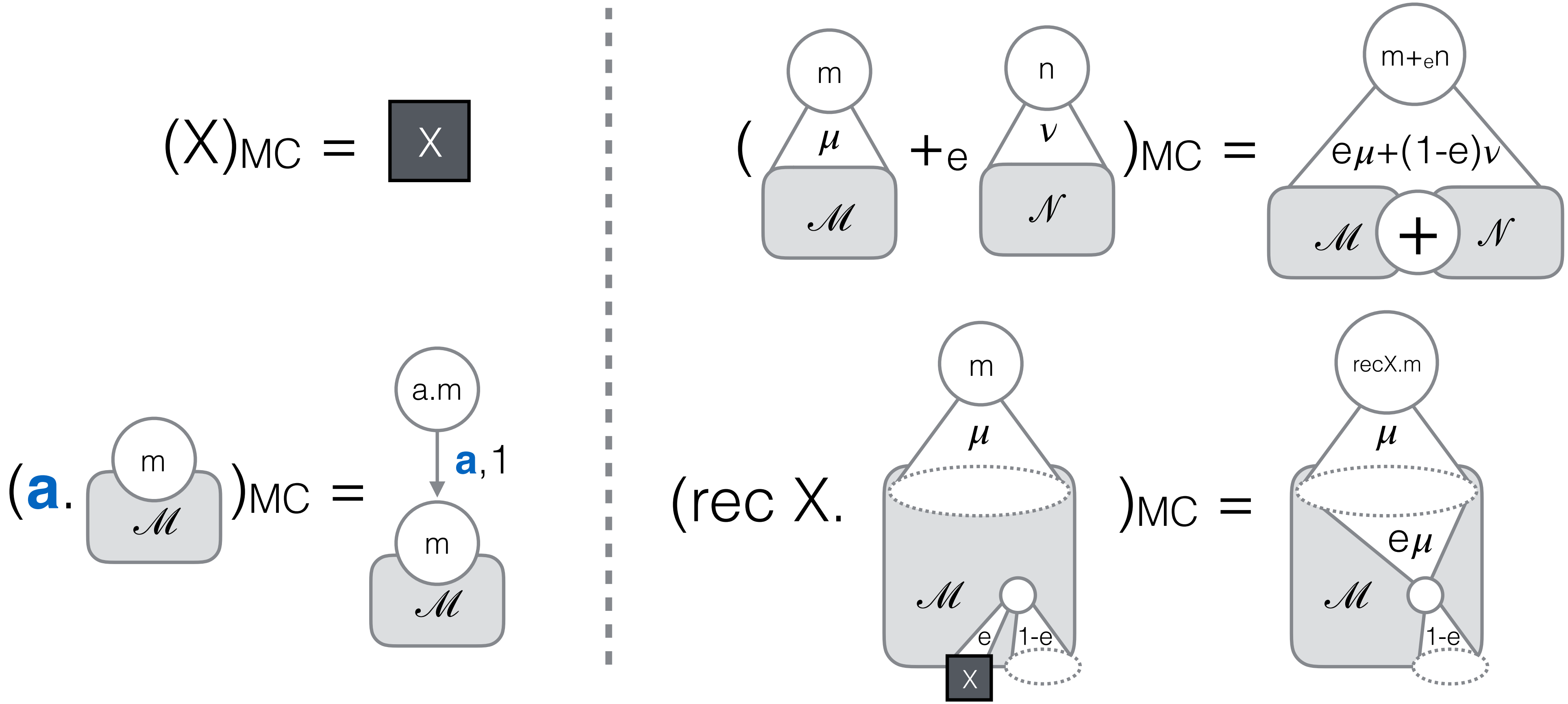
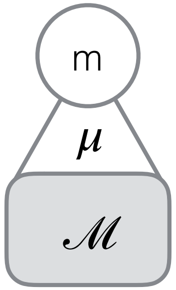
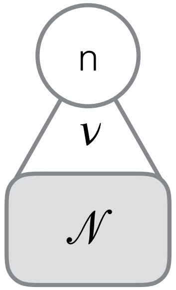


Bisimilarity distance for MCs

(Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

$d_{MC}($ 

, 

$) = \min \{ \int \Lambda(d_{MC}) \, d\omega \mid \omega \in \Omega(\mu, \nu) \}$

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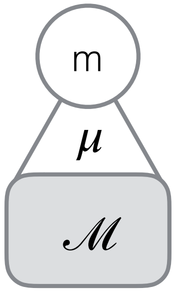
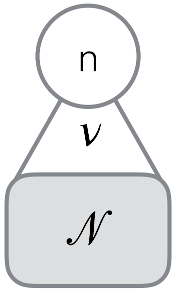
$d_{MC}($ 

, 

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$\underbrace{\min \{ \int \Lambda(d_{MC})\, d\omega \mid \omega \in \Omega(\mu,\nu) \}}_{\text{Kantorovich lifting}}$

Kantorovich lifting

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couplings
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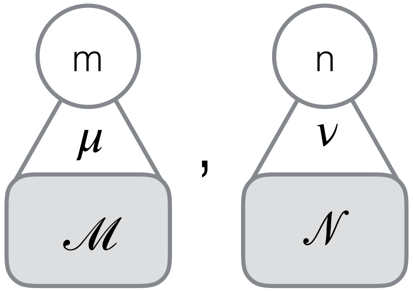

$$d_{MC}(m, n) = \min \{ \int \Lambda(d_{MC})\, d\omega \mid \omega \in \Omega(\mu,\nu) \}$$

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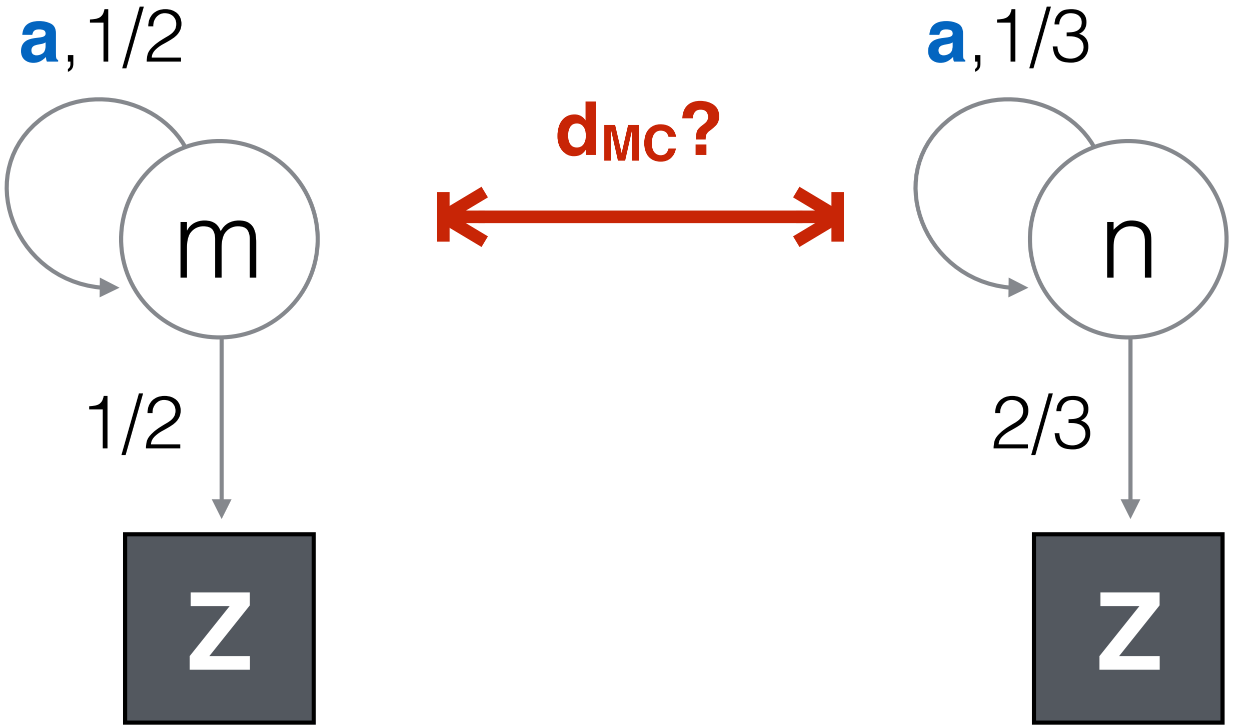
$d_{MC}(\, m, \mu, \mathcal{M} \,,\, n, \nu, \mathcal{N}\,) = \min \{ \int \Lambda(d_{MC})\, d\omega \mid \omega \in \Omega(\mu,\nu) \}$

Kantorovich lifting

$\Lambda(d_{MC})$ — greatest 1-bounded pseudometric on $(\mathbb{A} \times MC) \cup \mathbb{X}$

s.t, for all $\mathbf{a} \in \mathbb{A}$, $\Lambda(d_{MC})((\mathbf{a}, m\ \mathcal{M}),(\mathbf{a}, n\ \mathcal{N})) = d_{MC}(m\ \mathcal{M}, n\ \mathcal{N})$

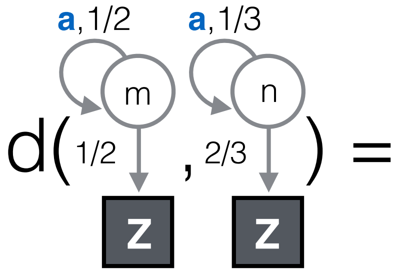
Running example



m = rec X. (**a**.X $+_{1/2}$ Z)   n = rec Y. (**a**.Y $+_{1/3}$ Z)

optimal coupling between
transition probabilities
of m and n

ω^*	$\nu((\mathbf{a},n)) = \mathbf{1/3}$	$\nu(Z) = \mathbf{2/3}$
$\mu((\mathbf{a},m))=\mathbf{1/2}$	1/3	1/6
$\mu(Z)=\mathbf{1/2}$		1/2



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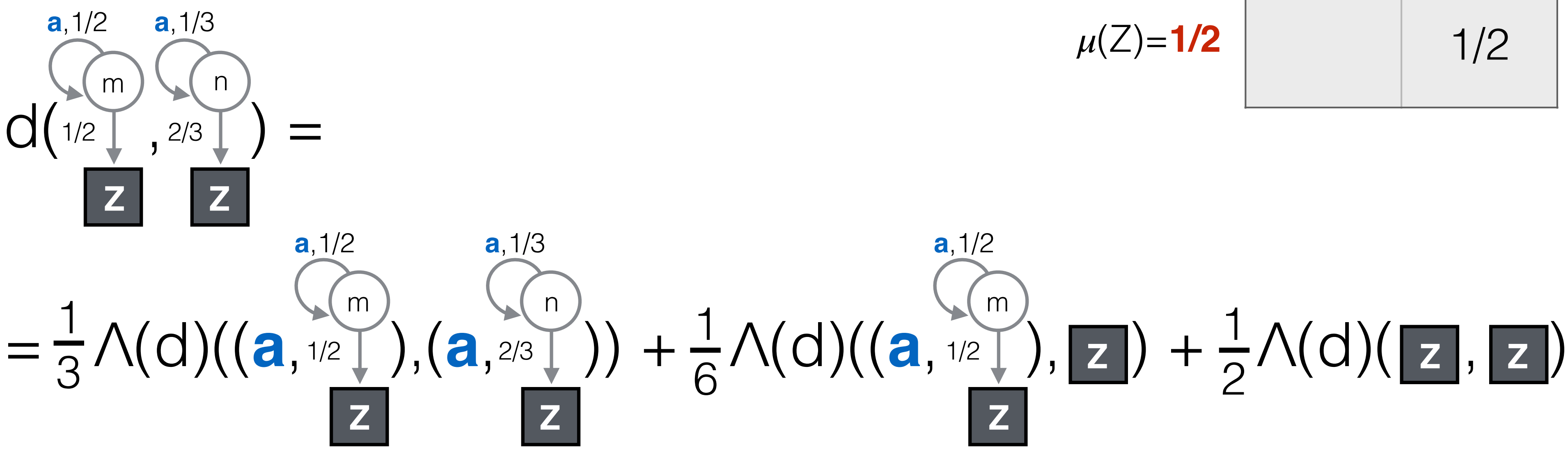

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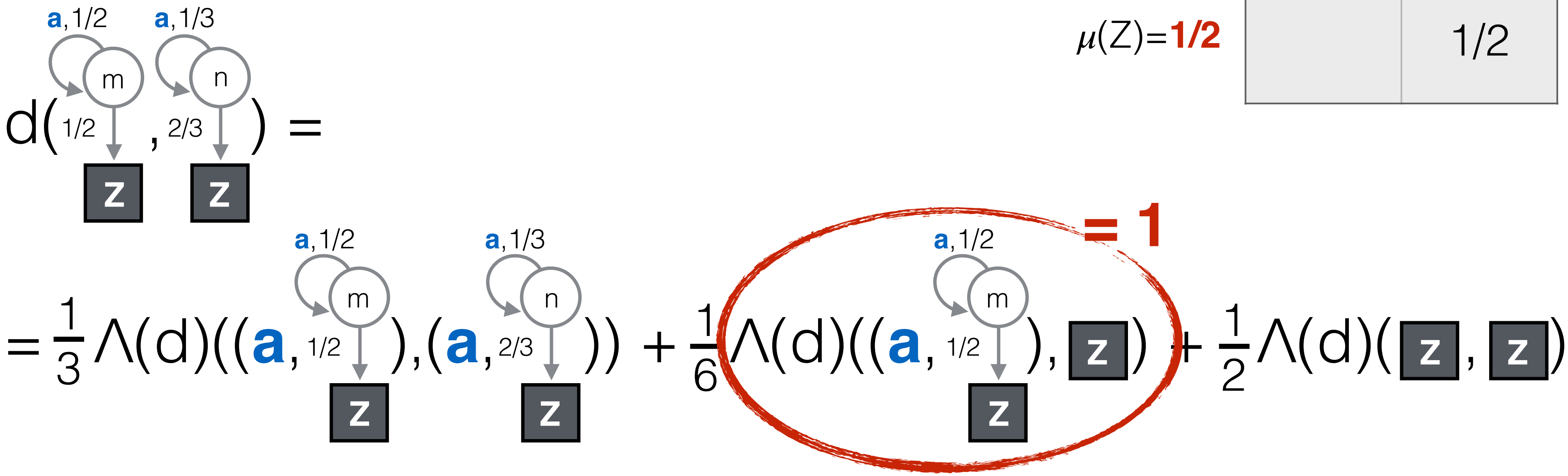

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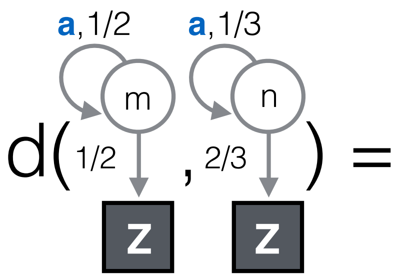

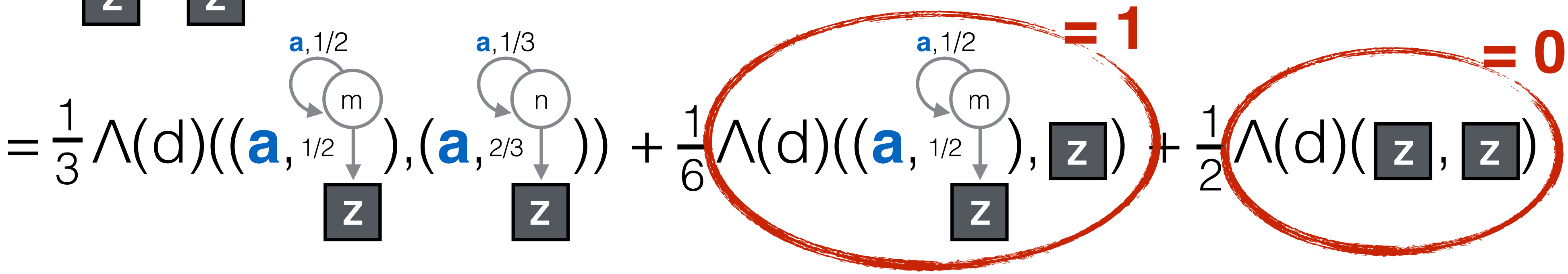

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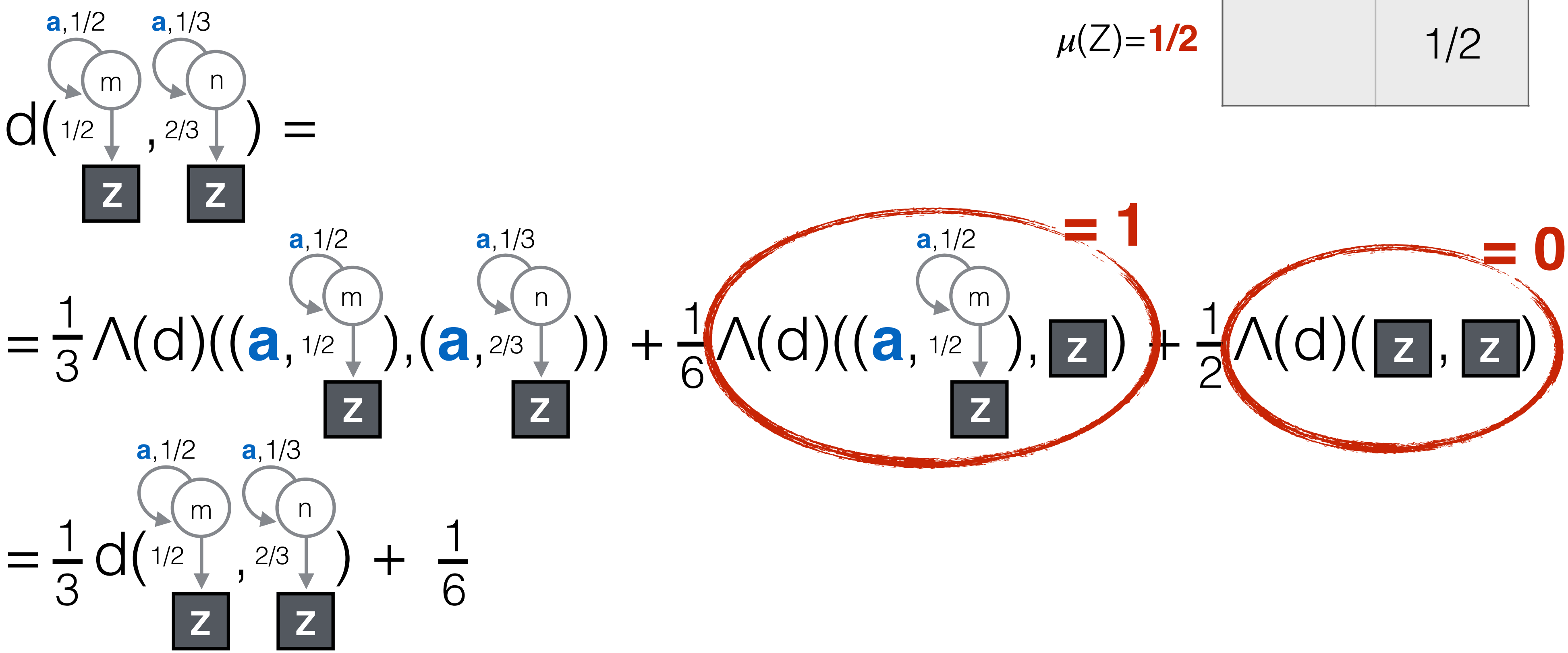

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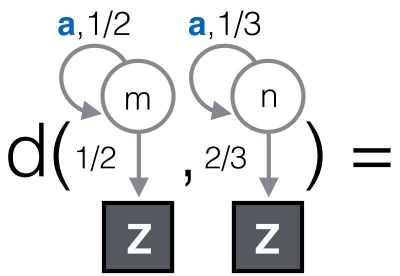

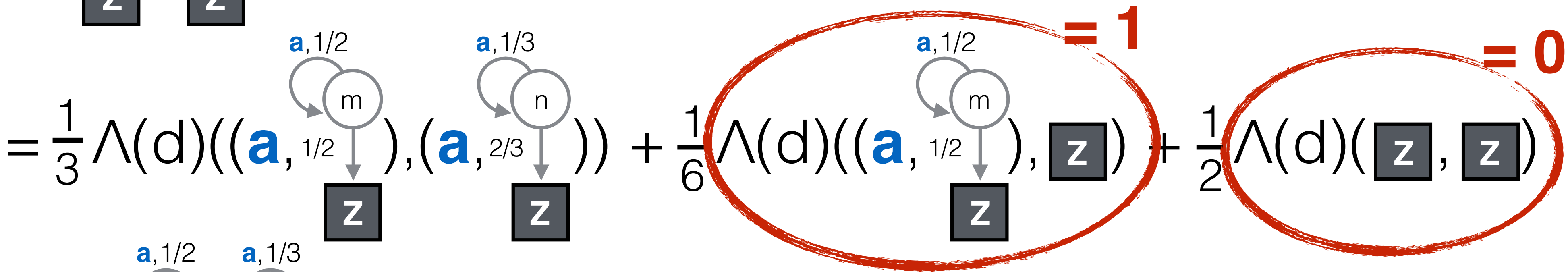

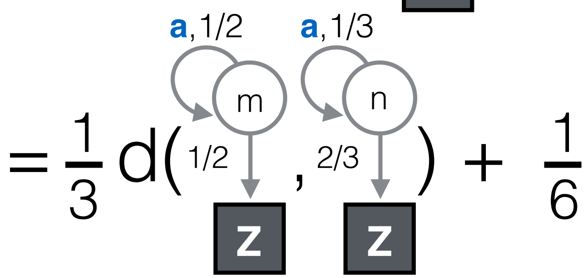

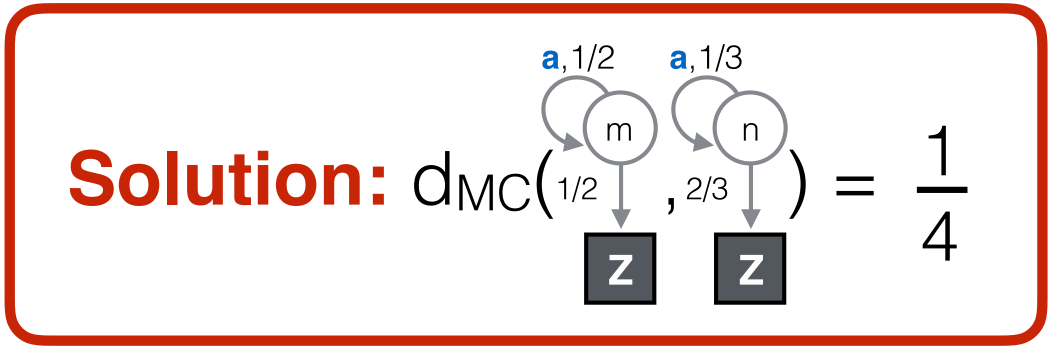

The Quantitative Equational Theory

Axiomatization (first attempt)

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(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$ — for $\delta \leq e\varepsilon+(1-e)\varepsilon'$

(Top) $\vdash t =_1 s$

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Axiomatization (first attempt)

Interpolative barycentric axioms
(Mardare-Panangaden-Plotkin LICS'16)

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— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

the terms from the example…

$m = \text{rec } X.\ (\mathbf{a}.X +_{1/2} Z)$ $n = \text{rec } Y.\ (\mathbf{a}.Y +_{1/3} Z)$

(IB) $\{t =_{\varepsilon} s,\ t' =_{\varepsilon'} s'\} \vdash t +_{e} t' =_{\delta} s +_{e} s'$

— for $\delta \le e\varepsilon + (1\text{-}e)\varepsilon'$

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$m = \text{rec } X.\ (\mathbf{a}.X +_{1/2} Z)$ $n = \text{rec } Y.\ (\mathbf{a}.Y +_{1/3} Z)$

$\mathbf{a}.X +_{1/2} Z =_{0} (\mathbf{a}.X +_{1/3} \mathbf{a}.X) +_{1/2} Z$ (B2)

$=_{0} \mathbf{a}.X +_{\mathbf{1/6}} (\mathbf{a}.X +_{2/5} Z)$ (SA)

(IB) $\{t =_{\varepsilon} s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_{\delta} s +_e s'$

— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

the terms from the example…

$m = \text{rec } X. (\mathbf{a}.X +_{1/2} Z)$ $n = \text{rec } Y. (\mathbf{a}.Y +_{1/3} Z)$

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rec is problematic…

The quantitative equational framework
of Mardare-Panangaden-Plotkin requires
all operators to be **non-expansive**

(NExp) $\{t_1 =_\varepsilon s_1, \ldots, t_n =_\varepsilon s_n\} \vdash f(t_1, \ldots, t_n) =_\varepsilon f(s_1, \ldots s_n)$ — for $f \in \Sigma$

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… but the NExp axiom is not sound for recursion

$$\mathscr{A}_{MC} \not\models \big(\{t =_\varepsilon s\} \vdash \text{rec } X.t =_\varepsilon \text{rec } X.s\big)$$

(see Gebler-Larsen-Tini FoSSaCS'15)

Relaxing non-expansivity

we keep all the axioms of quantitative algebras
but the NExp axiom

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

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the Archimedian axiom will be used to recover completeness

from what we have seen in the example before and (Fix)+(Unfold)+(Top)+(IB) we obtain

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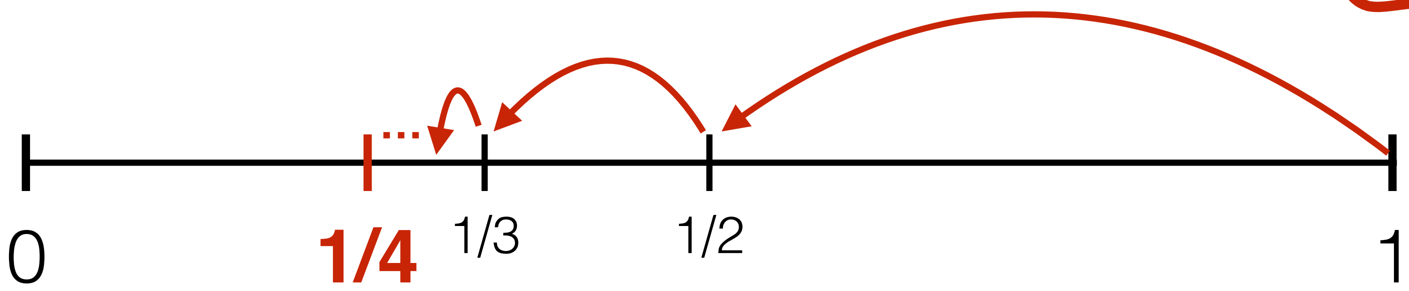

$(\vdash m =_{\varepsilon} n)$

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$\Rightarrow \vdash m =_{1/4} n$

Sound & Complete Axiomatization

Interpolative barycentric axioms
(Mardare-Panangaden-Plotkin LICS'16)

(B1) $\vdash t +_1 s =_0 t$

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash t +_e s =_0 s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u =_0 t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e,e' \in [0,1)$

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$ — for $\delta \le e\varepsilon + (1-e)\varepsilon'$

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Milner's recursion axioms

(Unfold) $\vdash \text{rec } X.t = t[\text{rec } X.t / X]$

(Fix) $\{t = s[t / X]\} \vdash t = \text{rec } X.s$ — for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) = \text{rec } X.t$

(Cong) $\{t =_0 s\} \vdash \text{rec } X.t =_0 \text{rec } X.s$

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A quantitative Kleene's theorem

$$(MC/\sim, d_{MC})$$

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$d_{\vdash}([t],[s]) = \inf\{\ \varepsilon \mid\ \vdash t =_{\varepsilon} s\ \}$

A quantitative Kleene's theorem

$$(\mathrm{MC}/\!\sim,\ d_{\mathrm{MC}})$$

$$\cong$$

isometric isomorphism

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Conclusions

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future work…

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Conclusions

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future work…

- What about different models? (e.g., non-determinism)
- What about different notions of distances?
- Beyond non-expansive operators

Thank you
for your attention