

Complete Axiomatization for the Bisimilarity Distance on MCs

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Introduction

- **Kleene's Theorem:** fundamental correspondence between regular expressions and DFAs
- **Salomaa'66, Kozen'91:** complete axiomatization for proving equivalence of regular expressions
- **Milner'84:** applied the above program on process behaviors and LTSs
- Many variations of the above schema

Example: Markov chains

Expressions: $t, s ::= X \mid a.t \mid t +_e s \mid \text{rec } X.t$

Example: Markov chains

names
 $X \in \mathbb{X}$

Expressions: $t, s := X \mid a.t \mid t +_e s \mid \text{rec } X.t$

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The expression is defined as $t, s ::= X \mid a.t \mid t +_e s \mid \text{rec } X.t$. The term X is associated with the label "names" and $X \in \mathbb{X}$. The term $a.t$ is associated with the label "action-prefix" and $a \in \mathbb{A}$.

Example: Markov chains

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The expression $t, s ::= X \mid a.t \mid t +_e s \mid \text{rec } X.t$ is shown. Three callout boxes explain the components:

- A box labeled "names $X \in \mathbb{X}$ " points to the variable X .
- A box labeled "probabilistic choice" points to the term $a.t$.
- A box labeled "action-prefix $a \in \mathbb{A}$ " points to the term $t +_e s$.

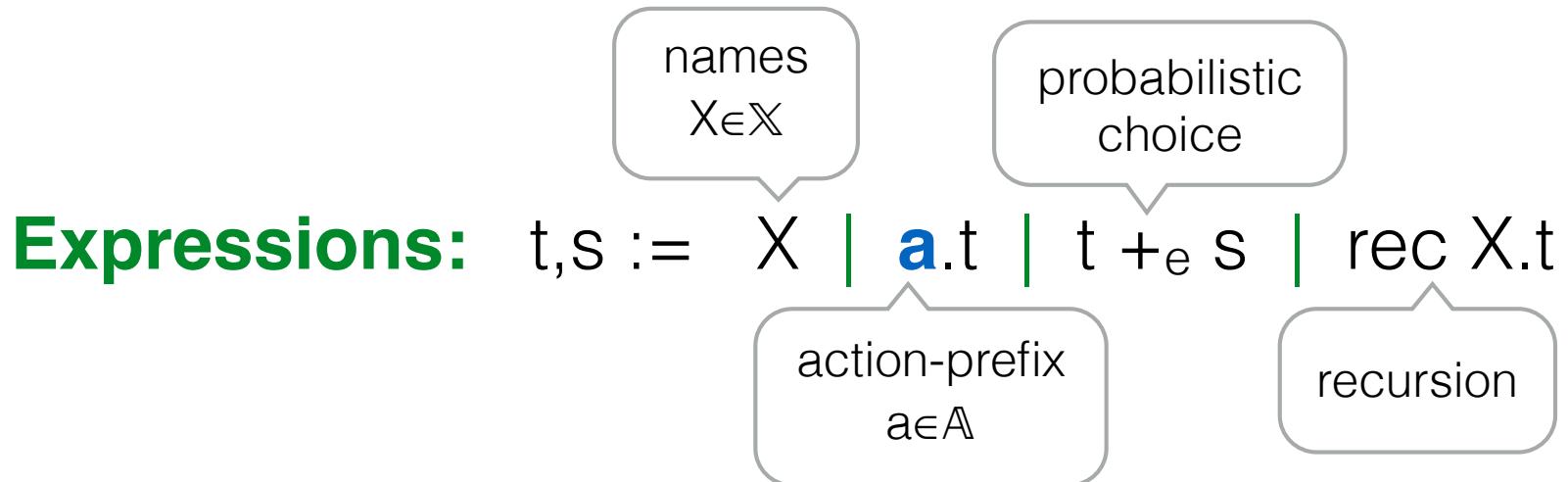
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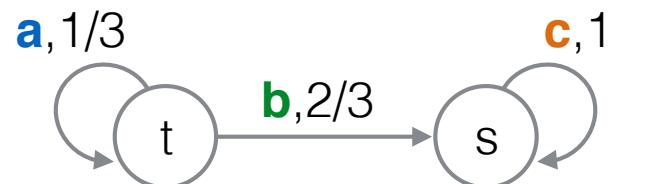
The diagram illustrates the components of the expression grammar. It shows four main categories in rounded rectangles with arrows pointing to specific parts of the grammar:

- "names $X \in \mathbb{X}$ " points to the variable X .
- "probabilistic choice" points to the action prefix $a.t$.
- "action-prefix $a \in \mathbb{A}$ " points to the action prefix $a.t$.
- "recursion" points to the recursion operator $\text{rec } X.t$.

Example: Markov chains



Kleene's theorem for MCs



$$\begin{aligned}t &= \text{rec } X.(\mathbf{a}.X +_{1/3} \mathbf{b}.s) \\s &= \text{rec } Y.(\mathbf{c}.Y)\end{aligned}$$

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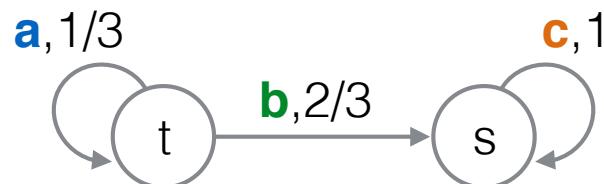
probabilistic
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action-prefix
 $a \in \mathbb{A}$

recursion

finite MCs

Kleene's theorem for MCs



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Example: Markov chains

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e, e' \in [0, 1]$

(Unfold) $\vdash \text{rec } X.t = t[\text{rec } X.t / X]$

(Fix) $\{t = s[t / X]\} \vdash t = \text{rec } X.s$ — for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) = \text{rec } X.t$

Example: Markov chains

Stark-Smolka
axiomatization

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Stone's barycentric axioms

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Milner's recursion axioms

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...for probabilistic systems

- **Generative Markov chains:**
Baeten-Bergstra-Smolka‘95 & Stark-Smolka‘00
- **Simple Probabilistic Automata:**
Bandini-Segala‘01
- **(fully) Probabilistic Automata:**
Mislove-Ouaknine-Worrell‘04 (strong-bisimulation)
Deng-Palamidessi‘07 (weak-bisimulation & behavioral eq.)
- **Quantitative Kleene Coalgebras:**
Silva-Bonchi-Bonsangue-Rutten‘11 (coagebraic bisim.)

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By using **Quantitative Equational Theories*** of
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$$s = t \quad \longrightarrow \quad s =_{\varepsilon} t$$

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completeness
almost for free!

$$s =_{\varepsilon} t$$

Equational Theories

$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

Equational Theories

$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

(Refl) $\vdash t = t$

(Symm) $\{t = s\} \vdash s = t$

(Trans) $\{t = u, u = s\} \vdash t = s$

(Cong) $\{t_1 = s_1, \dots, t_n = s_n\} \vdash f(t_1, \dots, t_n) = f(s_1, \dots, s_n)$ — for $f \in \Sigma$

Quantitative Theories

Mardare-Panangaden-Plotkin (LICS'16)

$$\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s$$

quantitative
inference

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_{\varepsilon} s\} \vdash s =_{\varepsilon} t$

(Triang) $\{t =_{\varepsilon} u, u =_{\delta} s\} \vdash t =_{\varepsilon+\delta} s$

(NExp) $\{t_1 =_{\varepsilon} s_1, \dots, t_n =_{\varepsilon} s_n\} \vdash f(t_1, \dots, t_n) =_{\varepsilon} f(s_1, \dots, s_n) \quad \text{for } f \in \Sigma$

(Max) $\{t =_{\varepsilon} s\} \vdash t =_{\varepsilon+\delta} s \quad \text{for } \delta > 0$

(Arch) $\{t =_{\delta} s \mid \delta > \varepsilon\} \vdash t =_{\varepsilon} s$

Quantitative Semantics

Quantitative Algebra

$$\mathcal{A} = (A, \Sigma_A, d_A) \begin{array}{l} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \begin{array}{l} (A, \Sigma_A) \text{ — Universal algebra} \\ (A, d_A) \text{ — metric space} \end{array}$$

Satisfiability

$$\mathcal{A} \models \left(\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s \right)$$

iff

for all $i \in I$. $d_A([t_i], [s_i]) \leq \varepsilon_i$ implies $d_A([t], [s]) \leq \varepsilon$

completeness

quantitative
algebra

$$\mathcal{A} \models (\vdash t =_{\varepsilon} s) \quad (\vdash t =_{\varepsilon} s) \in \mathcal{U}$$

quantitative
theory

soundness

quantitative
algebra

completeness

quantitative
theory

$$\mathcal{A}_{MC} \models (\vdash t =_{\varepsilon} s) \quad (\vdash t =_{\varepsilon} s) \in \mathcal{U}_{MC}$$

soundness

The Quantitative Universal Algebra

Universal Algebra of MCs

Signature: $X : 0 \mid a.- : 1 \mid +_e : 2 \mid \text{rec } X : 1$

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$$(X)_{\text{MC}} = \boxed{X}$$

$$(a. \boxed{m})_{\text{MC}} = \begin{array}{c} a.m \\ \downarrow \\ m \end{array}$$

Universal Algebra of MCs

Signature: $X : 0 \mid a.- : 1 \mid +_e : 2 \mid \text{rec } X : 1$

$$(X)_{\text{MC}} = \boxed{X}$$

$$(a. \begin{array}{c} m \\ \mathcal{M} \end{array})_{\text{MC}} = \begin{array}{c} a.m \\ \downarrow \\ \begin{array}{c} m \\ \mathcal{M} \end{array} \end{array}$$

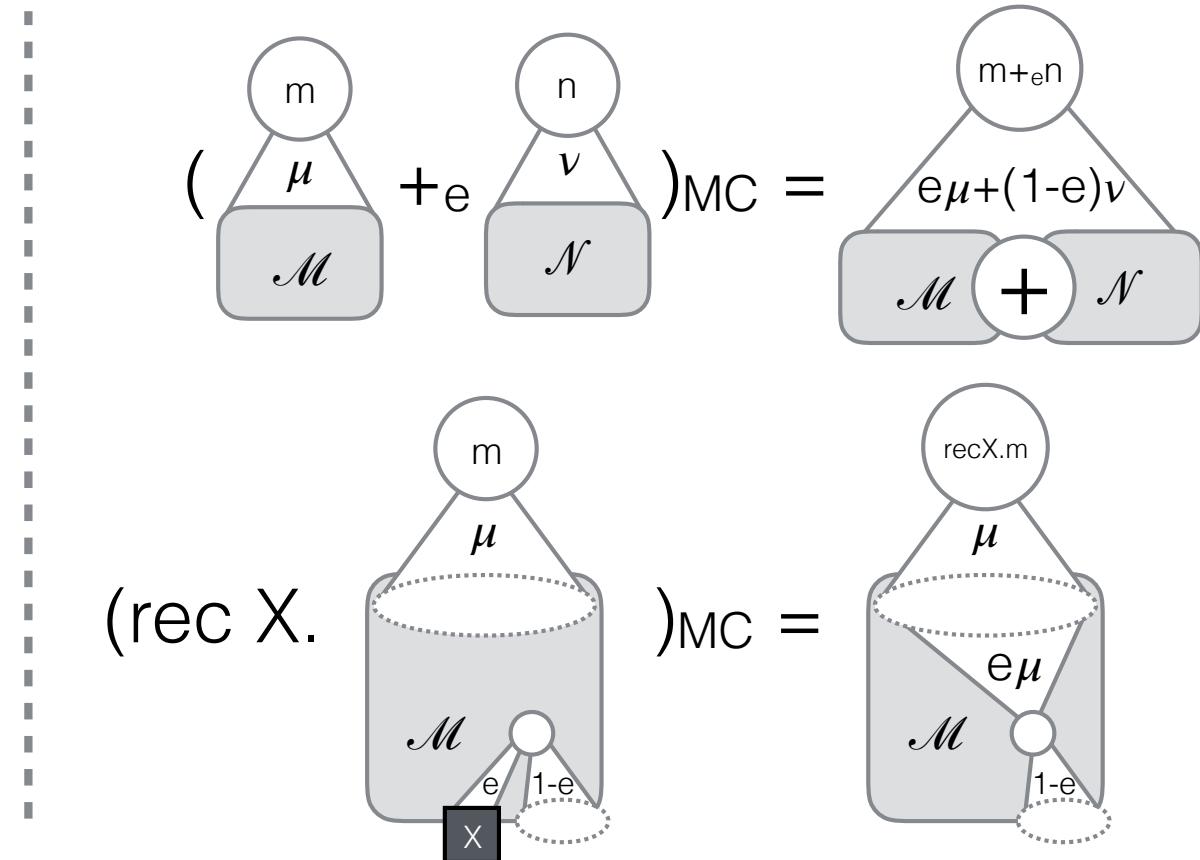
$$(\begin{array}{c} m \\ \mu \\ \mathcal{M} \end{array} +_e \begin{array}{c} n \\ \nu \\ \mathcal{N} \end{array})_{\text{MC}} = \begin{array}{c} m+n \\ e\mu+(1-e)\nu \\ \mathcal{M} + \mathcal{N} \end{array}$$

Universal Algebra of MCs

Signature: $X : 0 \mid a.- : 1 \mid +_e : 2 \mid \text{rec } X : 1$

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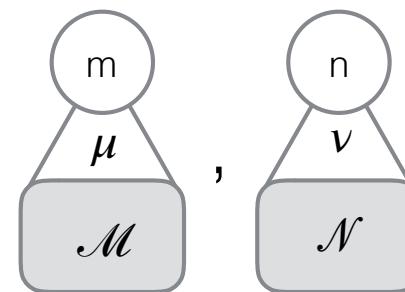
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Bisimilarity distance for MCs

(Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

$$d_{MC}(\mu, \nu) = \min \{ \int \Lambda(d_{MC}) d\omega \mid \omega \in \Omega(\mu, \nu) \}$$


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Kantorovich lifting

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couplings
= probabilistic “relations”

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Kantorovich lifting

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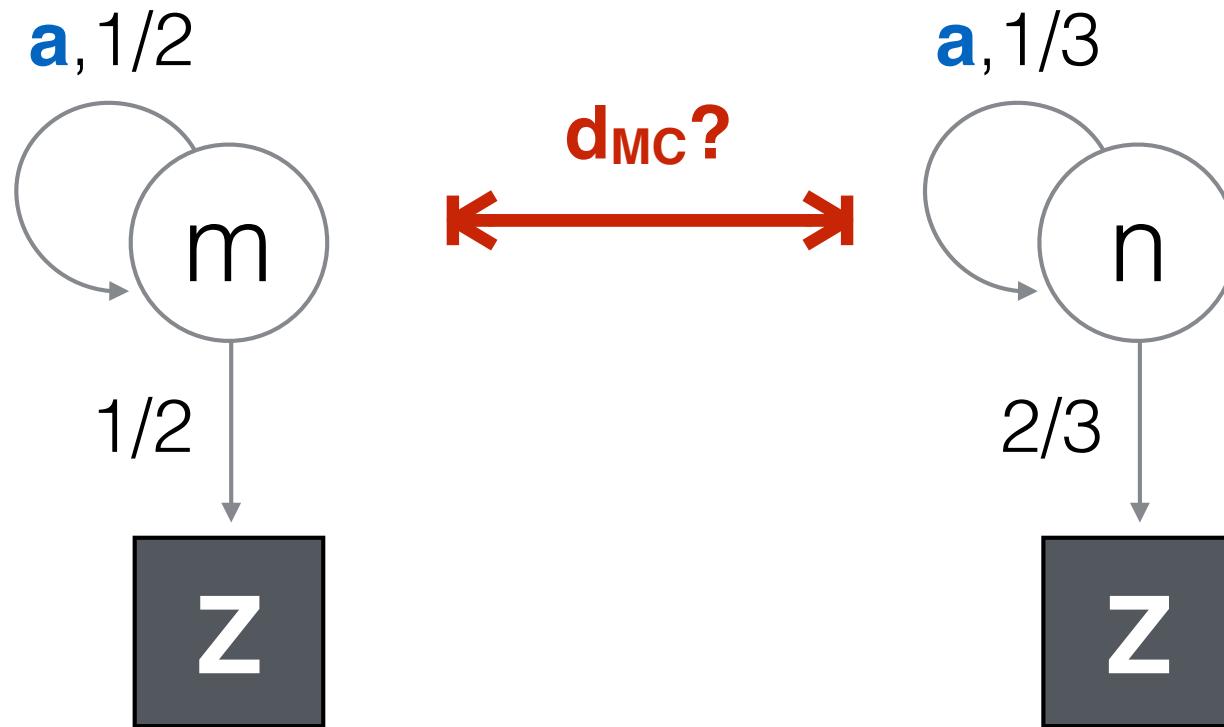
The diagram shows two states, μ and ν , each represented by a rounded rectangle containing a smaller circle labeled m and n respectively. Between them is a horizontal line with a vertical bar, representing the coupling set $\Omega(\mu, \nu)$. A red arrow points from this coupling set towards the integral expression in the equation above. A green curved arrow points from the same expression towards the text "Kantorovich lifting". A callout bubble contains the text "couplings = probabilistic ‘relations’".

$\Lambda(d_{MC})$ — greatest 1-bounded pseudometric on $(\mathbb{A} \times MC) \cup \mathbb{X}$

s.t, for all $a \in \mathbb{A}$, $\Lambda(d_{MC})((a, \mu), (a, \nu)) = d_{MC}(\mu, \nu)$

The diagram shows two states, μ and ν , each represented by a rounded rectangle containing a smaller circle labeled m and n respectively. A blue arrow points from a state a to the first term in the equation, indicating the condition for the greatest 1-bounded pseudometric.

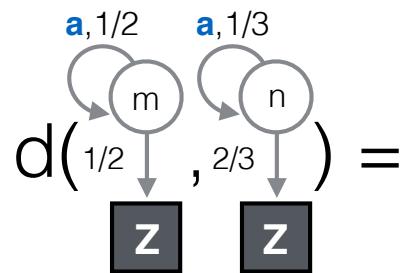
Running example



$$m = \text{rec } X. (a.X +_{1/2} Z)$$

$$n = \text{rec } Y. (a.Y +_{1/3} Z)$$

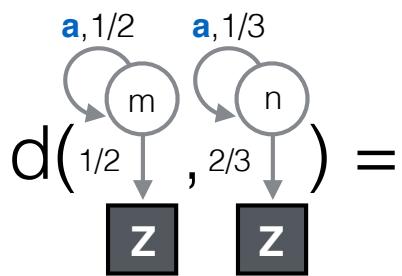
optimal coupling between
transition probabilities
of m and n



ω^*

$\nu((\mathbf{a},n))$	$\nu(Z)$
$\frac{1}{3}$	$\frac{2}{3}$
$\mu((\mathbf{a},m)) = 1/2$	$\mu(Z) = 1/2$
$\frac{1}{3}$	$\frac{1}{6}$
	$\frac{1}{2}$

optimal coupling between
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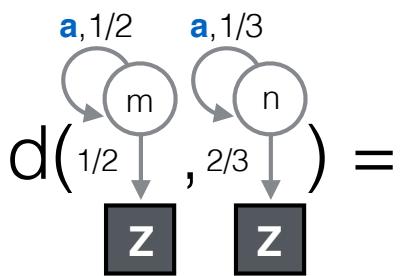
$$= \frac{1}{3} \wedge(d)((\mathbf{a}, \begin{matrix} a,1/2 \\ 1/2 \end{matrix}), (\mathbf{a}, \begin{matrix} a,1/3 \\ 2/3 \end{matrix})) + \frac{1}{6} \wedge(d)((\mathbf{a}, \begin{matrix} a,1/2 \\ 1/2 \end{matrix}), [\mathbf{z}]) + \frac{1}{2} \wedge(d)([\mathbf{z}], [\mathbf{z}])$$

$$\mu((\mathbf{a},m)) = \mathbf{1/2}$$

$$\mu(Z) = \mathbf{1/2}$$

$\nu((\mathbf{a},n))$	$\nu(Z)$
$\mathbf{1/3}$	$\mathbf{2/3}$
1/3	1/6
	1/2

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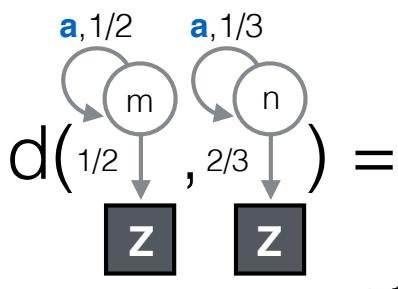


$$= \frac{1}{3} \Lambda(d)((\text{a}, \begin{array}{c} 1/2 \\ 1/2 \end{array}), (\text{a}, \begin{array}{c} 1/3 \\ 2/3 \end{array})) + \frac{1}{6} \Lambda(d)((\text{a}, \begin{array}{c} 1/2 \\ 1/2 \end{array}), [z]) + \frac{1}{2} \Lambda(d)([z], [z])$$

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ω^*

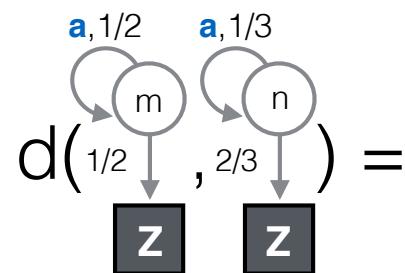
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optimal coupling between
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$$\omega^* \quad \begin{array}{c} \nu((\mathbf{a},n)) \\ \parallel \\ 1/3 \end{array} \quad \begin{array}{c} \nu(Z) \\ \parallel \\ 2/3 \end{array}$$

$$\mu((\mathbf{a},m)) = \mathbf{1/2}$$



$$= \frac{1}{3} \Lambda(d)((\mathbf{a}, \underset{1/2}{\mathbf{a}}), (\mathbf{a}, \underset{2/3}{\mathbf{a}})) + \frac{1}{6} \Lambda(d)((\mathbf{a}, \underset{1/2}{\mathbf{a}}), [\mathbf{z}]) + \frac{1}{2} \Lambda(d)([\mathbf{z}], [\mathbf{z}])$$

$$= \frac{1}{3} d(\underset{1/2}{\mathbf{a}}, \underset{2/3}{\mathbf{a}}) + \frac{1}{6}$$

= 1

= 0

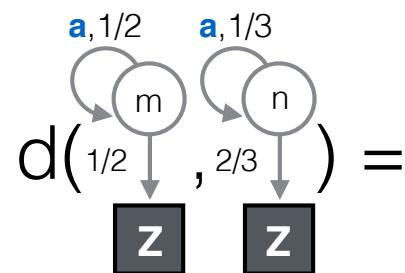
optimal coupling between
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$$\omega^* \quad \begin{matrix} \nu((\mathbf{a},n)) \\ \parallel \\ 1/3 \end{matrix} \quad \begin{matrix} \nu(Z) \\ \parallel \\ 2/3 \end{matrix}$$

$$\mu((\mathbf{a},m)) = 1/2$$

$$\mu(Z) = 1/2$$

$\nu((\mathbf{a},n))$	$\nu(Z)$
1/3	1/6
	1/2



$$= \frac{1}{3} \Lambda(d)((\mathbf{a}, \underset{1/2}{\mathbf{a}}), (\mathbf{a}, \underset{2/3}{\mathbf{a}})) + \frac{1}{6} \Lambda(d)((\mathbf{a}, \underset{1/2}{\mathbf{a}}), [z]) + \frac{1}{2} \Lambda(d)([z], [z])$$

$$= \frac{1}{3} d(\underset{1/2}{\mathbf{a}}, \underset{2/3}{\mathbf{a}}) + \frac{1}{6}$$

Solution: $d_{MC}(\underset{1/2}{\mathbf{a}}, \underset{2/3}{\mathbf{a}}) = \frac{1}{4}$

The Quantitative Equational Theory

Axiomatization (first attempt)

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(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$ — for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

(Top) $\vdash t =_1 s$

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Axiomatization (first attempt)

Interpolative barycentric axioms

(B1) $\vdash t +_1 s =_0 t$

(Mardare-Panangaden-Plotkin LICS'16)

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash t +_e s =_0 s +_{1-e} t$

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— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

the terms from the example...

$m = \text{rec } X. (\mathbf{a}.X +_{1/2} Z)$ $n = \text{rec } Y. (\mathbf{a}.Y +_{1/3} Z)$

(IB) $\{t =_{\varepsilon} s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_{\delta} s +_e s'$
— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

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$$m = \text{rec } X. (\mathbf{a}.X +_{1/2} Z) \quad n = \text{rec } Y. (\mathbf{a}.Y +_{1/3} Z)$$

$$\begin{aligned} \mathbf{a}.X +_{1/2} Z &=_0 (\mathbf{a}.X +_{1/3} \mathbf{a}.X) +_{1/2} Z && \text{(B2)} \\ &=_0 \mathbf{a}.X +_{1/6} (\mathbf{a}.X +_{2/5} Z) && \text{(SA)} \end{aligned}$$

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rec is problematic...

The quantitative equational framework
of Mardare-Panangaden-Plotkin requires
all operators to be **non-expansive**

(NExp) $\{t_1 =_\varepsilon s_1, \dots, t_n =_\varepsilon s_n\} \vdash f(t_1, \dots, t_n) =_\varepsilon f(s_1, \dots, s_n)$ — for $f \in \Sigma$

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... but the NExp axiom is not sound for recursion

$\mathcal{A}_{MC} \not\models (\{t =_\varepsilon s\} \vdash \text{rec } X.t =_\varepsilon \text{rec } X.s)$

(see Gebler-Larsen-Tini FoSSaCS'15)

Relaxing non-expansivity

we keep all the axioms of quantitative algebras
but the NExp axiom

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(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

(NExp) $\{t_1 =_\varepsilon s_1, \dots, t_n =_\varepsilon s_n\} \vdash f(t_1, \dots, t_n) =_\varepsilon f(s_1, \dots, s_n)$ — for $f \in \Sigma$

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the Archimedean axiom will be used
to recover completeness

from what we have seen in the example before and
(Fix)+(Unfold)+(Top)+(IB) we obtain

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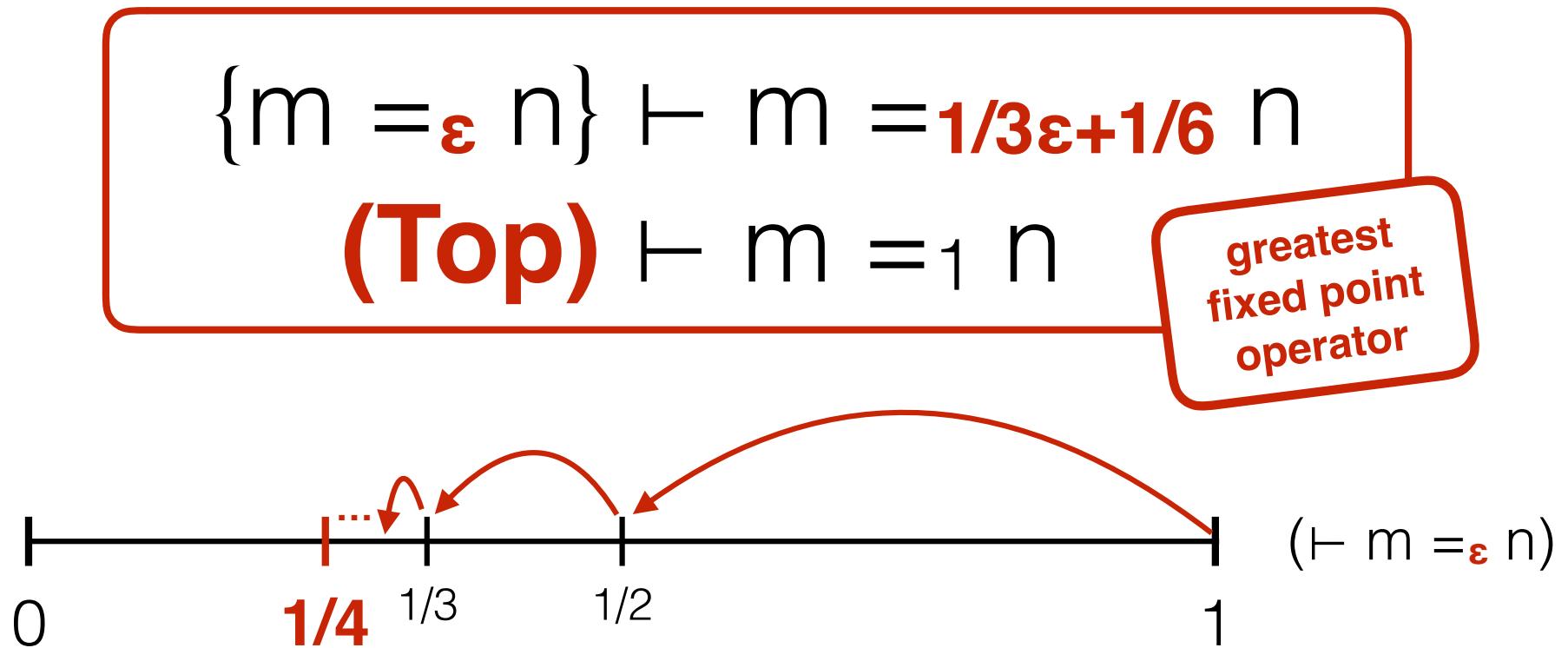
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$$(\textbf{Top}) \vdash m =_1 n$$

greatest
fixed point
operator

from what we have seen in the example before and
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$\rightarrow \vdash m =_{1/4} n$

Sound & Complete Axiomatization

Interpolative barycentric axioms

(Mardare-Panangaden-Plotkin LICS'16)

$$(B1) \vdash t +_1 s =_0 t$$

$$(B2) \vdash t +_e t =_0 t$$

$$(SC) \vdash t +_e s =_0 s +_{1-e} t$$

$$(SA) \vdash (t +_e s) +_{e'} u =_0 t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u) \quad \text{--- for } e, e' \in [0, 1]$$

$$(IB) \{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s' \quad \text{--- for } \delta \leq e\varepsilon + (1-e)\varepsilon'$$

$$(Top) \vdash t =_1 s$$

Milner's recursion axioms

$$(Unfold) \vdash \text{rec } X.t = t[\text{rec } X.t / X]$$

$$(Fix) \{t = s[t / X]\} \vdash t = \text{rec } X.s \quad \text{--- for } X \text{ guarded in } t$$

$$(Unguard) \vdash \text{rec } X.(t +_e X) = \text{rec } X.t$$

$$(Cong) \{t =_0 s\} \vdash \text{rec } X.t =_0 \text{rec } X.s$$

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A quantitative Kleene's theorem

$(MC/\sim, d_{MC})$

$(Exp/=, d_{\vdash})$

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$$d_\vdash([t], [s]) = \inf\{ \varepsilon \mid \vdash t =_\varepsilon s \}$$

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\mathbb{R}

isometric
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Conclusions

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- Sound&Complete Axiomatization
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future work...

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future work...

- What about different models? (e.g., non-determinism)
- What about different notions of distances?
- Beyond non-expansive operators

Thank you
for your attention