Quantitative Algebraic Reasoning

(an Overview)

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(invited talk, **EXPRESS/SOS 2020**)

based on joint work with
R. Mardare, P. Panangaden, G. Plotkin

Why Algebraic Reasoning?

Algebraic reasoning is extensively used in process algebras
...actually, all theoretical computer science!

- **Representability:** *describing mathematical structures as an algebra of operations subject to equations*
  eg: Labelled transition systems as process algebras

- **Effectful Languages:** understanding computational effects as operations to be performed during execution
  eg: Moggi's monadic effects

- **Algorithms:** *compositional reasoning, up-to techniques, normal forms, etc...*

An Application

Consider the recursively defined CCS process $P \overset{\text{def}}{=} a \mid P$

not image finite!

$$P \sim P \mid P$$

Bisimulation **up-to technique** is an elegant and way to prove it!

$$P \sim a \mid P$$
$$x \sim 0 \mid x$$
$$(x \mid y) \mid z \sim x \mid (y \mid z)$$
$$x \mid y \sim y \mid x$$

$$\begin{tikzcd}
P \arrow[r, dash, "\mathscr{R}"] \arrow[d, "a"'] & P \mid P \arrow[d, "a"] \\
a^n \mid 0 \mid P \sim P \arrow[r, dash, "\mathscr{R}"] & P \mid P \sim a^p \mid P \mid 0 \mid a^q \mid P
\end{tikzcd}$$

To understand how should be a quantitative generalisation
of the concept of equational logic we first need to understand

Universal algebras

(its categorical generalisation)

Historical Perspective

- **Lawvere'64:** *categorically axiomatised (the clone of) equational theories*

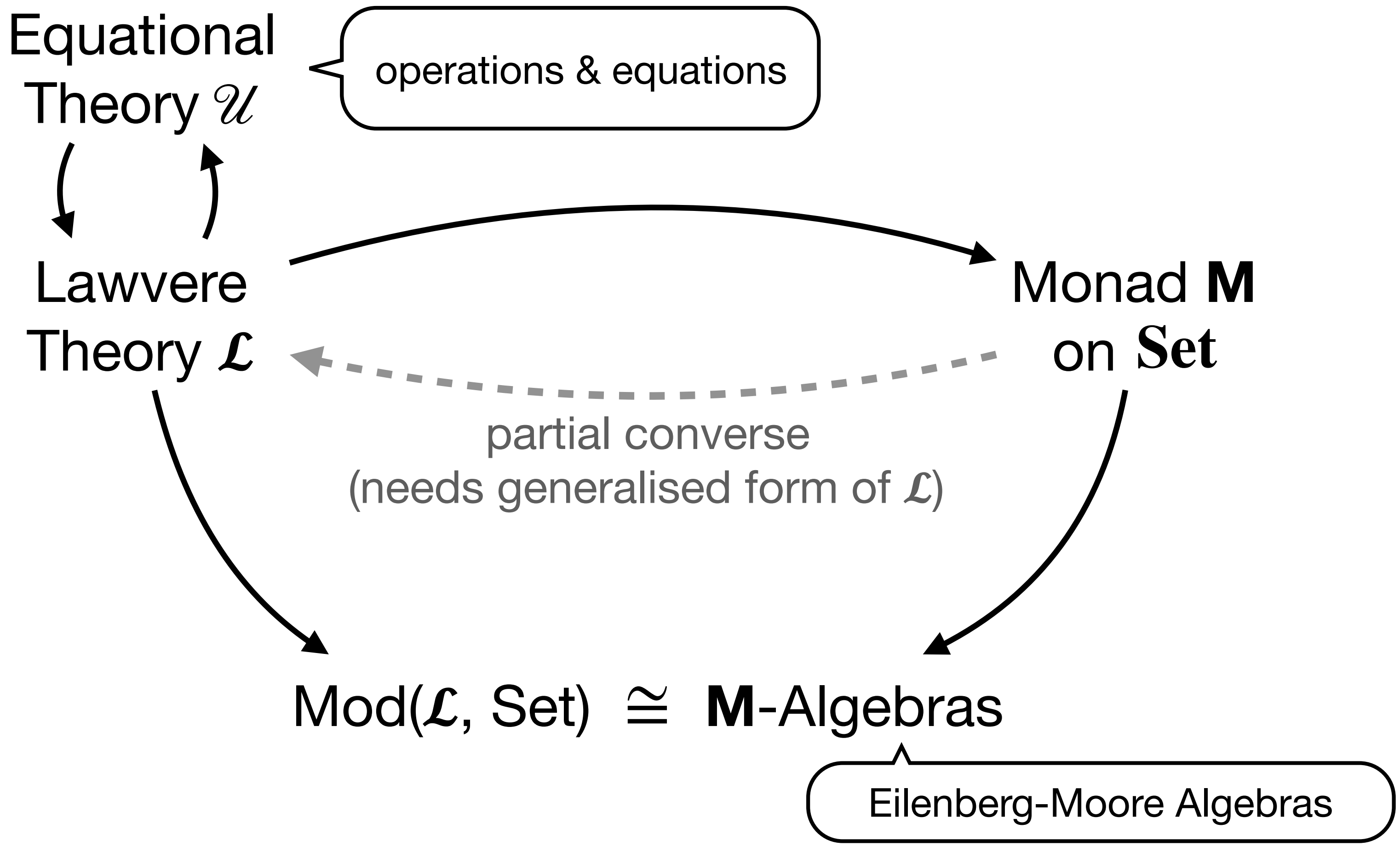
- **Eilenberg-Moore'65, Kleisli'65:** Every standard construction is induced by a pair of adjoint functors, and EM-algebras for are universal algebras

- **Linton'66:** general connection between monads and Lawvere theories

Universal Algebras

Lawvere'64, Linton'66

(the standard picture)



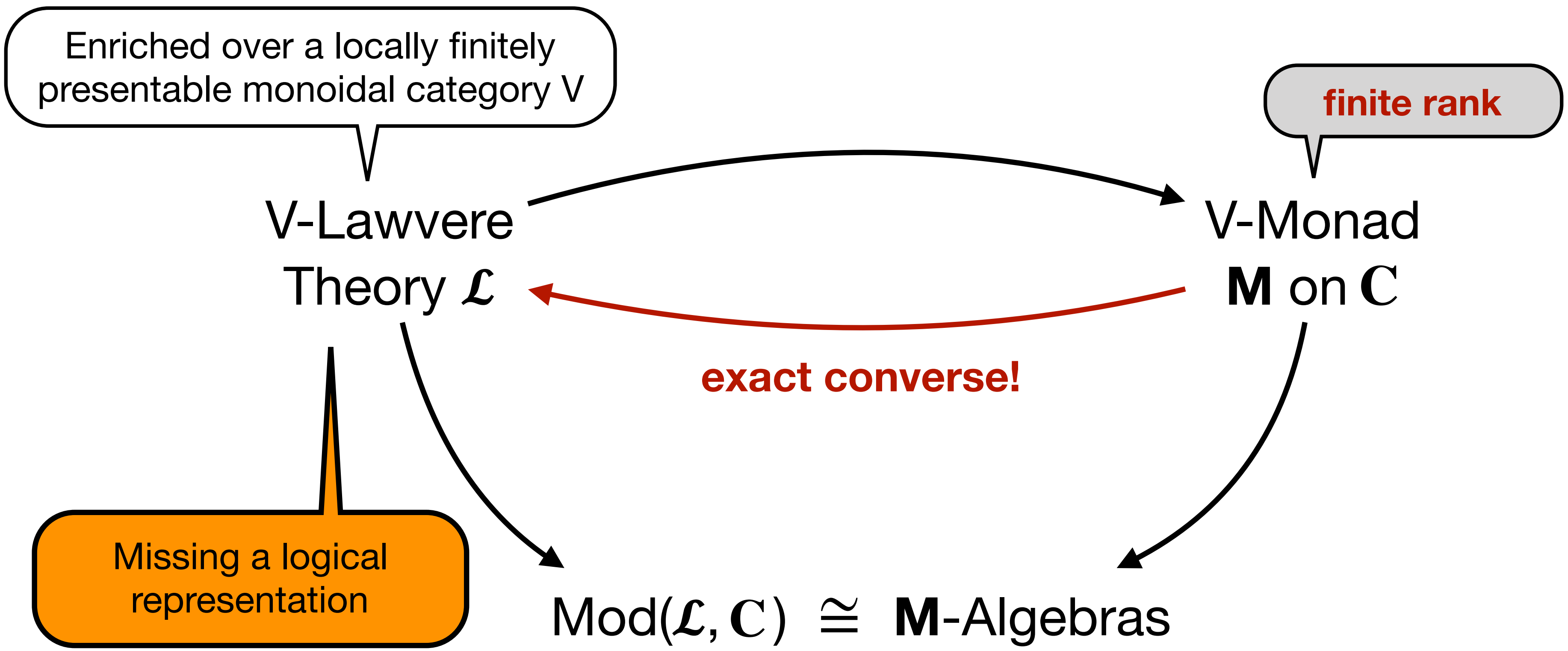
Historical Perspective
(continued)

- ...
- **Moggi'88:** *How to incorporate effects into denotational semantics?* - **Monads** as notions of computations
- **Plotkin & Power'01:** *(most of the) Monads are given by operations and equations* expressed by means of enriched Lawvere Theories - **Generic Algebraic Effects**

Universal Algebras

(the "enriched"picture)

Power (TAC'99)

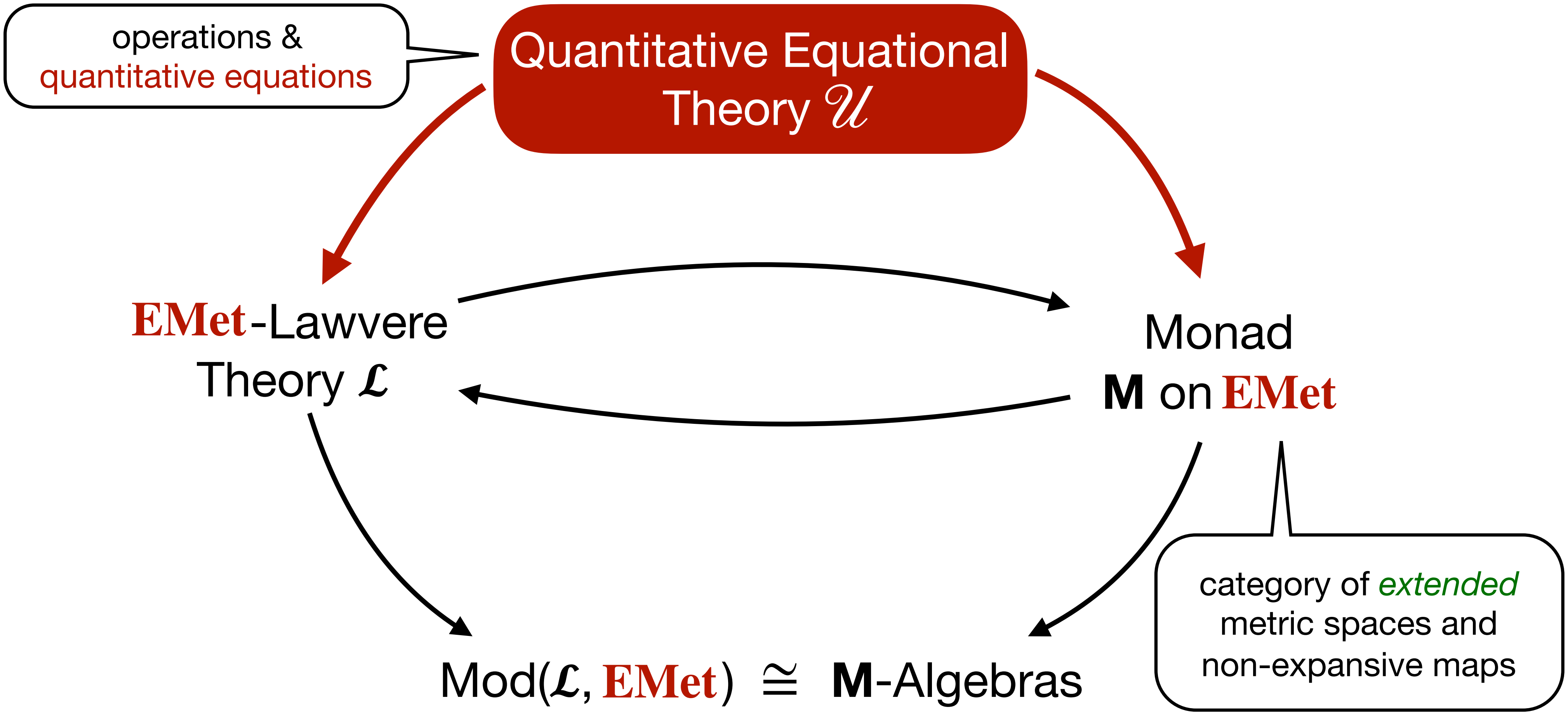


Historical Perspective

(continued)

- ...
- **Moggi'88:** *How to incorporate effects into denotational semantics?* - **Monads** as notions of computations
- **Plotkin & Power'01:** *(most of the) Monads are given by operations and equations* expressed by means of enriched Lawvere Theories - **Generic Algebraic Effects**
- **Mardare, Panangaden, Plotkin (LICS'16):**
  Theory of effects in a metric setting
  \- **Quantitative Algebraic Effects** (operations & *quantitative equations* give monads on **EMet**)

The picture of the Talk



Quantitative Equations

$$s =_{\varepsilon} t$$

"s is approximately equal to t up to an error ε"

Quantitative Equational Theory

Mardare, Panangaden, Plotkin (LICS'16)

A quantitative equational theory $\mathscr{U}$ of type Σ is a set of

$$\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_\varepsilon s$$

conditional quantitative equations

closed under substitution of variables, logical inference, and the following "meta axioms"

(Refl) $\vdash x =_0 x$

(Symm) $x =_\varepsilon y \vdash y =_\varepsilon x$

(Triang) $x =_\varepsilon y, y =_\delta z \vdash x =_{\varepsilon+\delta} y$

(NExp) $x_1 =_\varepsilon y_1, \ldots, y_n =_\varepsilon y_n \vdash f(x_1, \ldots, x_n) =_\varepsilon f(y_1, \ldots, y_n)$ – **for** $f \in \Sigma$

(Max) $x =_\varepsilon y \vdash x =_{\varepsilon+\delta} y$ – **for** $\delta > 0$

(Inf) $\{x =_\delta y \mid \delta > \varepsilon\} \vdash x =_\varepsilon y$

Quantitative Algebras

Mardare, Panangaden, Plotkin (LICS'16)

The models of a quantitative equational theory $\mathscr{U}$ of type Σ are

Quantitative Σ-Algebras:

$\mathscr{A} = (A, \alpha : \Sigma A \to A)$ – **Universal Σ-algebras on EMet**

Satisfying the all the conditional quantitative equations in $\mathscr{U}$

Satisfiability

$$\mathscr{A} \models \left(\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s \right)$$

iff

for any assignment $\iota : X \to A$

$\left(\forall i \in I \,.\, d_A(\iota(t_i), \iota(s_i)) \leq \varepsilon_i \right)$ **implies** $d_A(\iota(t), \iota(s)) \leq \varepsilon$

Category of Models

The collection of models for $\mathscr{U}$ forms a category denoted by

$$\mathbb{K}(\Sigma, \mathscr{U})$$

with morphisms being the non-expansive homomorphisms

$$\begin{tikzcd} \Sigma A \arrow[r, "\alpha"] \arrow[d, "\Sigma h"'] & A \arrow[d, "h"] \\ \Sigma B \arrow[r, "\beta"] & B \end{tikzcd}$$

non-expansive
Σ-homomorphism

Quantitative Semilattices with $\perp$

Are the quantitative algebras over the signature

$$\Sigma_{\mathcal{S}} = \{\underbrace{\mathbf{0}}_{\text{bottom}} : 0, \underbrace{+}_{\text{join}} : 2\}$$

satisfying the following conditional quantitative equations

(S0) $\vdash x + \mathbf{0} =_0 x$

(S1) $\vdash x + x =_0 x$

(S2) $\vdash x + y =_0 y + x$

(S3) $\vdash (x + y) + z =_0 x + (y + z)$

(S4) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x + x' =_\delta y + y'$, **where** $\delta = \max\{\epsilon, \epsilon'\}$

Example of models

Unit interval with Euclidian distance and **max** as join

$([0,1], d_{[0,1]})$ $\quad (\mathbf{0})^{[0,1]} = 0$ $\quad (+)^{[0,1]} = \max$

Finite subsets with **Hausdorff distance**, with union as join

$(\mathscr{P}_{\textsf{fin}}(X), \mathscr{H}(d_X))$ $\quad (\mathbf{0})^{\mathscr{P}_{\textsf{fin}}} = \varnothing$ $\quad (+)^{\mathscr{P}_{\textsf{fin}}} = \cup$

Compact subsets with **Hausdorff distance**, with union as join

$(\mathscr{C}(X), \mathscr{H}(d_X))$ $\quad (\mathbf{0})^{\mathscr{C}} = \varnothing$ $\quad (+)^{\mathscr{C}} = \cup$

Interpolative Barycentric Algebras

Are the quantitative algebras over the signature

$$\Sigma_{\mathscr{B}} = \{ +_e : 2 \mid e \in [0,1] \}$$

convex sum

satisfying the following conditional quantitative equations

(B1) $\vdash x +_1 y =_0 x$

(B2) $\vdash x +_e x =_0 x$

(B3) $\vdash x + y =_0 y + x$

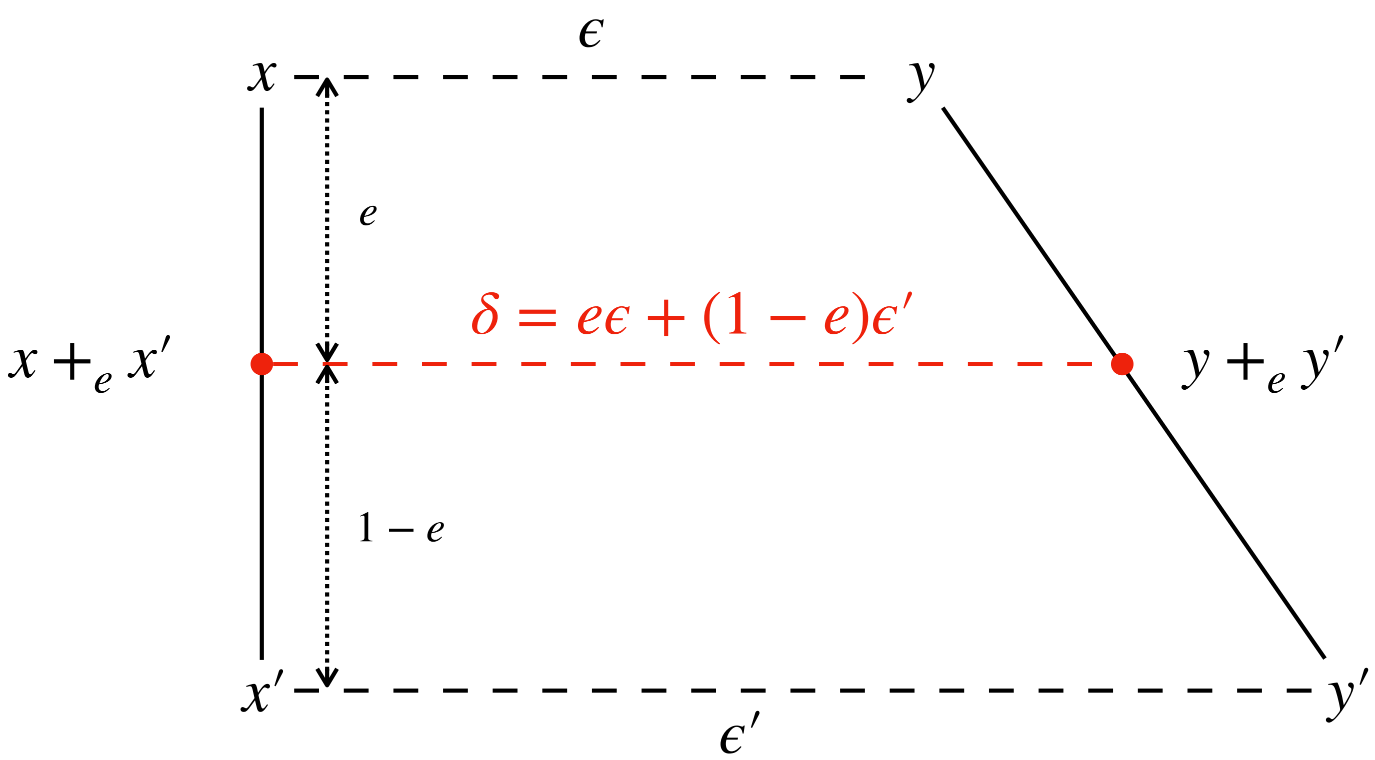
(SC) $\vdash x +_e y =_0 y +_{1-e} x$

(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{(1-e)e'}{1-ee'}} z)$, **for** $e, e' \in (0,1)$

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$

A geometric intuition

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$



Example of models

Unit interval with Euclidian distance and convex combinators

$$([0,1], d_{[0,1]}) \qquad (\ +_e\ )^{[0,1]}(a,b) = ea + (1-e)b$$

Finitely supported distributions with **Kantorovich distance**

$$(\mathscr{D}(X), \mathscr{K}(d_X)) \qquad (\ +_e\ )^{\mathscr{D}}(\mu,\nu) = e\mu + (1-e)\nu$$

Borel probability measures with **Kantorovich distance**

$$(\Delta(X), \mathscr{K}(d_X)) \qquad (\ +\ )^{\Delta}(\mu,\nu) = e\mu + (1-e)\nu$$

Completeness

Mardare, Panangaden, Plotkin (LICS'16)

For quantitative the equational logic we have an analogue of the usual completeness theorem

Theorem

$$\forall \mathscr{A} \in \mathbb{K}(\Sigma, \mathscr{U}) \,.\, \mathscr{A} \models (\Gamma \vdash t =_\varepsilon s)$$

Models of $\mathscr{U}$

iff

$$(\Gamma \vdash t =_\varepsilon s) \in \mathscr{U}$$

Free Quantitative Algebra

Given $\mathscr{U}$ and a metric space $M \in \mathbf{EMet}$,
there exists a quantitative algebra* $(T_M, \phi_M \colon \Sigma T_M \to T_M)$
and non-expansive map $\eta_M \colon M \to T_M$ such that

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[dr, "\beta"'] & T_M \arrow[d, dashed, "h"] & \Sigma T_M \arrow[l, "\phi_M"'] \arrow[d, "\Sigma h"] \\
& A & \Sigma A \in \mathbb{K}(\Sigma, \mathscr{U}) \arrow[l, "\alpha"]
\end{tikzcd}$$

Free-model for $\mathscr{U}$

(*) T_M is the term algebra with distance $d_M^{\mathscr{U}}(t, s) = \inf\{\epsilon \mid \vdash t =_\epsilon s \in \mathscr{U}_M\}$

Free Monad on $\mathbf{EMet}$

$$
\begin{tikzcd}
\mathbb{K}(\Sigma, \mathscr{U}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \\
\mathbf{EMet} \arrow[u, bend right]
\end{tikzcd}
$$

Models of $\mathscr{U}$ over extended metric spaces

Forgetful functor
(maps a quantitative algebra to its carrier)

Free QA functor
(maps a extended metric space to the free model of $\mathscr{U}$)

$$\Downarrow$$

$$T_{\mathscr{U}}$$

Monad induced by $\mathscr{U}$ on $\mathbf{EMet}$

$\mathscr{U}$ Models are $T_{\mathscr{U}}$-Algebras

Bacci, Mardare, Panangaden, Plotkin (LICS'18)

Theorem

For any ***basic*** quantitative equational theory $\mathscr{U}$ of type Σ

$$\mathbb{K}(\Sigma, \mathscr{U}) \cong T_{\mathscr{U}}\textbf{-Alg}$$

EM algebras for the monad $T_{\mathscr{U}}$

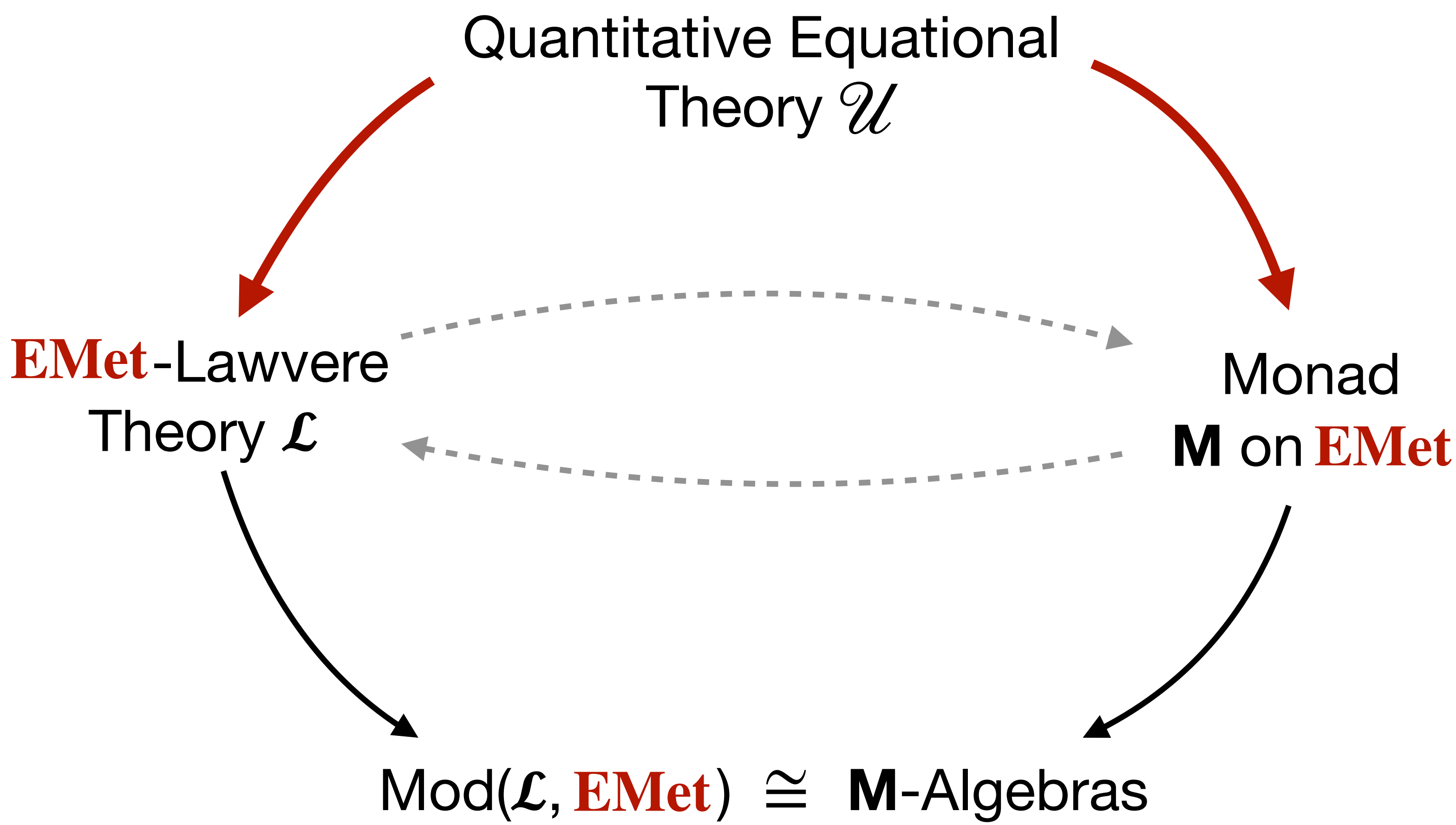
A quantitative equational theory $\mathscr{U}$ is *basic* if it can be axiomatised by a set of basic conditional quantitative equations

$$\{x_i =_{\varepsilon_i} y_i \mid i \in I\} \vdash t =_{\varepsilon} s$$

basic quantitative equation

The picture of the Talk

(...once again)



Examples of Monads

$$\begin{tikzcd} \mathbb{K}(\Sigma_{\mathscr{S}}, \mathscr{S}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \quad \Rightarrow \quad T_{\mathscr{S}} \cong \mathscr{P}_{\mathsf{fin}}$$

Quantitative semilattices with bottom

Finite subsets with **Hausdorff metric**

$$\begin{tikzcd} \mathbb{K}(\Sigma_{\mathscr{B}}, \mathscr{B}) \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathbf{EMet} \arrow[l, bend right] \end{tikzcd} \quad \Rightarrow \quad T_{\mathscr{B}} \cong \mathscr{D}$$

Interpolative barycentric algebras

Finitely supported probability distributions with **Kantorovich metric**

...and many more: ***total variation***, ***p-Wasserstein distance***, ***...***

The Continuous Case

(Complete Separable Metric Spaces)

all Cauchy
sequences have limit

exists a countable
dense subset

Free Monads on **CEMet**

A quantitative equational theory is *continuous* if it can be axiomatised by a collection of *continuous schemata* of quantitative equations

$$\{x_1 =_{\varepsilon_1} y_1, \ldots, x_n =_{\varepsilon_n} y_n\} \vdash t =_\varepsilon s \qquad \text{– for } \varepsilon \geq f(\varepsilon_1, \ldots, \varepsilon_n)$$

continuous real-valued function

$$\begin{tikzcd}
\mathbb{K}(\Sigma, \mathcal{U}) \arrow[rr, "\hat{\mathbb{C}}"] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & & \mathbb{C}\mathbb{K}(\Sigma, \mathcal{U}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \\
\mathbf{EMet} \arrow[rr, "\mathbb{C}"] \arrow[u, bend right] \arrow[d, Rightarrow] & & \mathbf{CEMet} \arrow[u, bend right] \arrow[d, Rightarrow] \\
T_{\mathcal{U}} & & \mathbb{C}T_{\mathcal{U}}
\end{tikzcd}$$

Models of $\mathcal{U}$ over *complete* metric spaces

Free Monads on $\mathbf{CSEMet}$

Models of $\mathscr{U}$ over metric spaces

Models of $\mathscr{U}$ over *complete* metric spaces

Complete separable metric spaces

$$
\begin{tikzcd}
\mathbb{K}(\Sigma, \mathscr{U}) \arrow[r, "\hat{\mathbb{C}}"] \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbb{C}\mathbb{K}(\Sigma, \mathscr{U}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \\
\mathbf{EMet} \arrow[u, bend right] \arrow[r, "\mathbb{C}"] \arrow[d, Rightarrow] & \mathbf{CEMet} \arrow[u, bend right] \arrow[d, Rightarrow] & \mathbf{CSEMet} \arrow[l, hook'] \arrow[d, Rightarrow] \\
T_{\mathscr{U}} & \mathbb{C}T_{\mathscr{U}} & \mathbb{C}T_{\mathscr{U}}
\end{tikzcd}
$$

If $\mathscr{U}$ is continuous and $T_{\mathscr{U}}$ preserves separability

Examples of Monads

$$\mathbb{CK}(\Sigma_{\mathscr{S}}, \mathscr{S}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{CEMet} \Rightarrow \mathbb{C}T_{\mathscr{S}} \cong \mathscr{P}_{\mathbf{C}}$$

Complete Quantitative semilattices with bottom

Compact subsets with **Hausdorff metric**

$$\mathbb{CK}(\Sigma_{\mathscr{B}}, \mathscr{B}) \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{CEMet} \Rightarrow \mathbb{C}T_{\mathscr{B}} \cong \Delta$$

Complete Interpolative barycentric algebras

Borel probability measures with **Kantorovich metric** (Giry monad)

...and many more: ***total variation***, ***p-Wasserstein distance***, ***...***

Combining Quantitative Theories

Disjoint Union of Theories

The disjoint union $\mathscr{U} + \mathscr{U}'$ of two quantitative theories with disjoint signatures is the smallest quantitative theory containing $\mathscr{U}$ and $\mathscr{U}'$

$$\begin{tikzcd}
\mathbb{K}(\Sigma, \mathscr{U}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbb{K}(\Sigma', \mathscr{U}') \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & & \mathbb{K}(\Sigma + \Sigma', \mathscr{U} + \mathscr{U}') \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \\
\mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] & \mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] & & \mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] \\
T_{\mathscr{U}} \arrow[r, phantom, "+"] & T_{\mathscr{U}'} \arrow[rr, phantom, "\overset{?}{\cong}"] & & T_{\mathscr{U}+\mathscr{U}'}
\end{tikzcd}$$

Models of $\mathscr{U} + \mathscr{U}'$

Disjoint Union of Theories

Bacci, Mardare, Panangaden, Plotkin (LICS'18)

The answer is positive for ***basic*** quantitative theories

$$T_{\mathscr{U}} + T_{\mathscr{U}'} \cong T_{\mathscr{U}+\mathscr{U}'}$$

The proof follows standard techniques (Kelly'80)

Theorem

For ***basic*** quantitative equational theories $\mathscr{U}, \mathscr{U}'$ of type Σ, Σ'

$$\mathbb{K}(\Sigma + \Sigma', \mathscr{U} + \mathscr{U}') \cong \langle T_{\mathscr{U}}, T_{\mathscr{U}'} \rangle\textbf{-Alg} \cong (T_{\mathscr{U}} + T_{\mathscr{U}'})\textbf{-Alg}$$

EM-**bialgebras** for the monads $T_{\mathscr{U}}, T_{\mathscr{U}'}$

Example: Markov chains

as the disjoint union of the theory of interpolative barycentric algebras with the theory of terminating executions with discount

Quantitative Theory of Terminating executions

$$\Sigma_{\mathscr{T}} = \{\underbrace{\mathbf{0}}_{\text{termination}}: 0, \underbrace{\diamond}_{\text{transition to next state}} : 1\}$$

(⋄-Lip) $\quad x =_{\epsilon} y \vdash \diamond x =_{\lambda\epsilon} \diamond y$

$$\begin{tikzcd}
\mathbb{K}(\Sigma_{\mathscr{B}}, \mathscr{B}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbb{K}(\Sigma_{\mathscr{T}}, \mathscr{T}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbb{K}(\Sigma_{\mathscr{B}} + \Sigma_{\mathscr{T}}, \mathscr{B} + \mathscr{T}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \\
\mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] & \mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] & \mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] \\
\mathscr{D} \quad + & \tilde{\Sigma}^{*}_{\mathscr{T}} \quad \cong & \mathscr{D} + \Sigma^{*}_{\mathscr{T}}
\end{tikzcd}$$

$$\mathscr{D} + \Sigma^{*}_{\mathscr{T}} \cong \mu y \,.\, \mathscr{D}(1 + \lambda \cdot y + -)$$

Acyclic finite Markov chains, with λ**-probabilistic bisimilarity metric**

...concretely

Acyclic finite Markov chains with **bisimilarity metric** are recovered as the free-algebra of the following quantitative equational theory

$$\Sigma_{\mathscr{B}} + \Sigma_{\mathscr{T}} = \{ \underbrace{+_e}_{\text{convex combination}} : 2 \mid e \in [0,1]\} \cup \{\underbrace{\mathbf{0}}_{\text{termination}} : 0, \underbrace{\diamond}_{\text{next state}} : 1\}$$

(B1) $\vdash x +_1 y =_0 x$

(B2) $\vdash x +_e x =_0 x$

(B3) $\vdash x + y =_0 y + x$

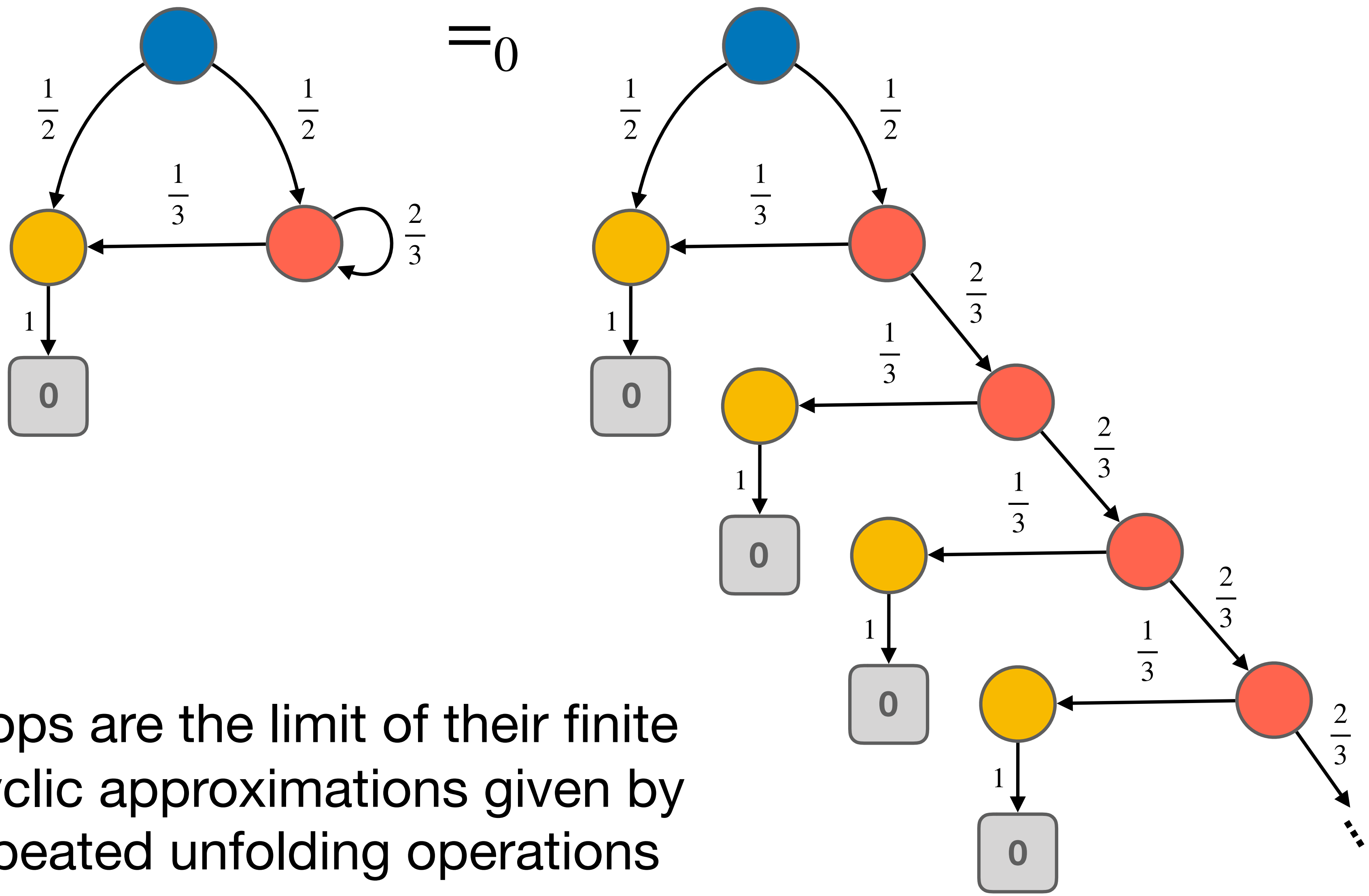
(SC) $\vdash x +_e y =_0 y +_{1-e} x$

(SA) $\vdash (x +_e y) +_{e'} z =_0 x +_{ee'} (y +_{\frac{(1-e)e'}{1-ee'}} z)$, **for** $e, e' \in (0,1)$

(IB) $x =_\epsilon y, x' =_{\epsilon'} y' \vdash x +_e x' =_\delta y +_e y'$, **where** $\delta = e\epsilon + (1-e)\epsilon'$

(◇-Lip) $x =_\epsilon y \vdash \diamond x =_{\lambda\epsilon} \diamond y$

What about loops?



Loops are the limit of their finite acyclic approximations given by repeated unfolding operations

Markov Processes

are the **completion** of the disjoint union of the theories of interpolative barycentric algebras with that of terminating executions with discount

$$
\begin{tikzcd}
\mathbb{CK}(\Sigma_{\mathscr{B}}, \mathscr{B}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbb{CK}(\Sigma_{\mathscr{T}}, \mathscr{T}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbb{CK}(\Sigma_{\mathscr{B}} + \Sigma_{\mathscr{T}}, \mathscr{B} + \mathscr{T}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \\
\mathbf{CEMet} \arrow[u, bend right] \arrow[d, Rightarrow] & \mathbf{CEMet} \arrow[u, bend right] \arrow[d, Rightarrow] & \mathbf{CEMet} \arrow[u, bend right] \arrow[d, Rightarrow] \\
\Delta \quad + & \tilde{\Sigma}^*_{\mathscr{T}} \quad \cong & \Delta + \Sigma^*_{\mathscr{T}} \cong \mu y \,.\, \Delta(1 + \lambda \cdot y + -\,)
\end{tikzcd}
$$

Markov processes on complete metric spaces with
λ-probabilistic bisimilarity metric

Final Coalgebra of MPs

$$\Delta + \Sigma^*_{\mathcal{T}} \cong \mu y\,.\,\Delta(1 + \lambda \cdot y + -\ )$$

assigns to any $A \in \mathbf{CSMet}$ the initial solution of the equation

$$MP_A \cong \Delta(1 + \lambda \cdot MP_A + A)$$

Theorem (Turi, Rutten'98)

Every *locally contractive functor* H on $\mathbf{CMet}$ has a unique fixed point, which is both an *initial algebra* and a *final coalgebra for* H

In particular, when $A \in \mathbf{0}$ (the empty metric space)

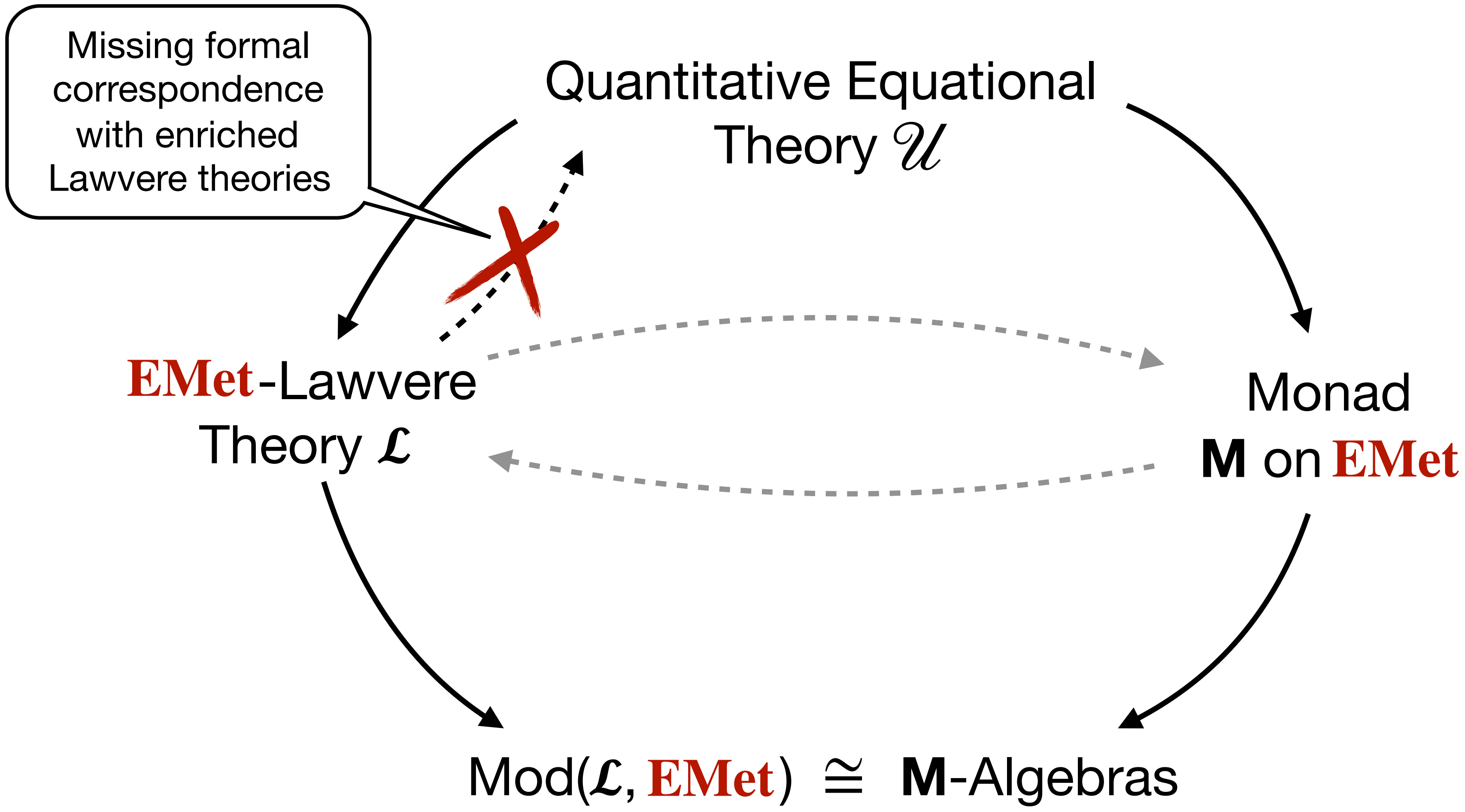
$$MP_{\mathbf{0}} \to \Delta(1 + \lambda \cdot MP_{\mathbf{0}})$$

final coalgebra of Markov processes

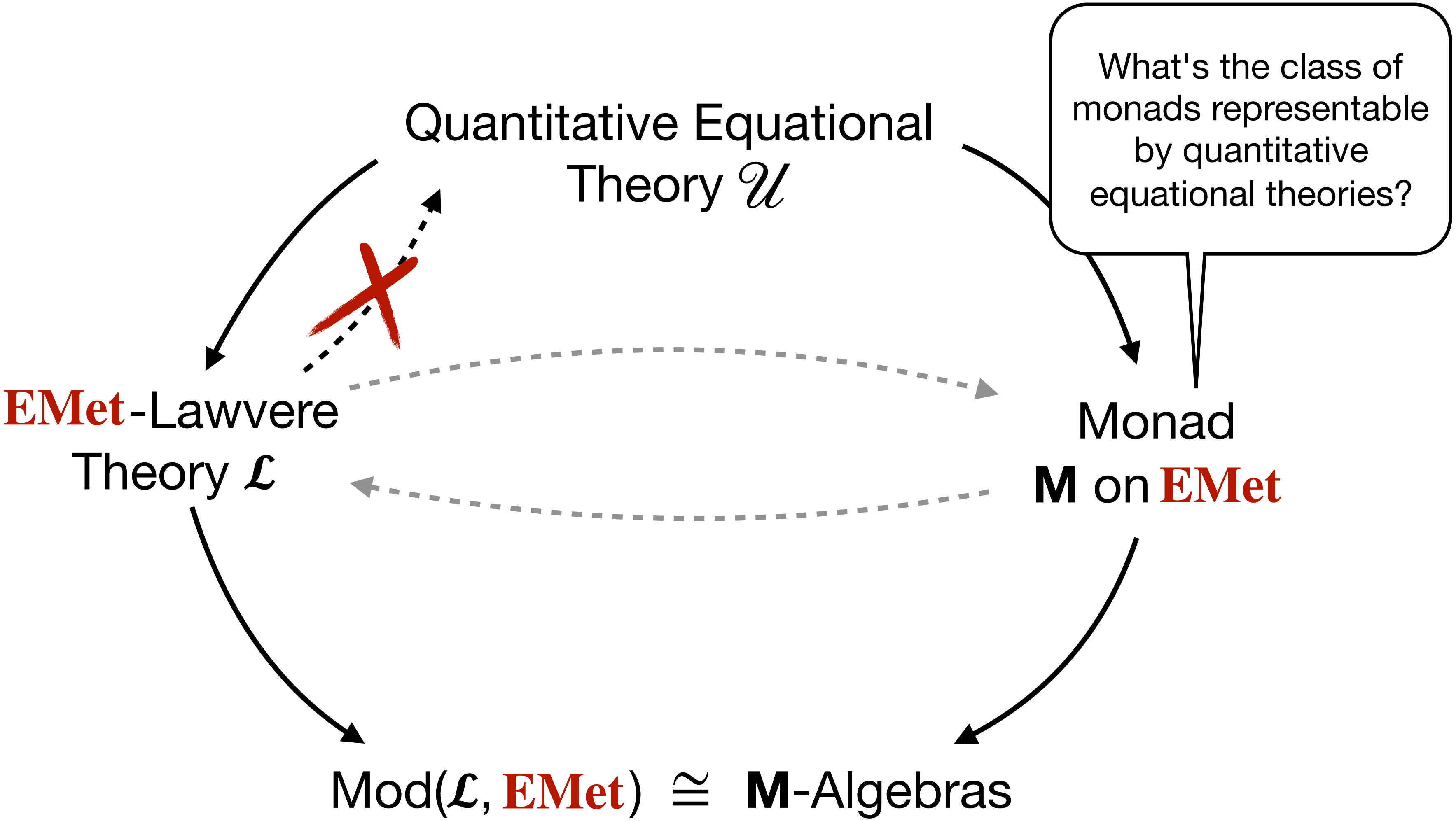
Open problems

(hence, future work!)

Open Problem 1



Open Problem 2



Open Problem 3

Currently we are exploring another way of combining quantitative equational theories:

$$
\begin{tikzcd}
\mathbb{K}(\Sigma, \mathscr{U}) \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & \mathbb{K}(\Sigma', \mathscr{U}') \arrow[d, bend right] \arrow[d, phantom, "\vdash"] & & \mathbb{K}(\Sigma + \Sigma', \mathscr{U} \otimes \mathscr{U}') \arrow[d, bend right] \arrow[d, phantom, "\vdash"] \\
\mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] & \mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] & & \mathbf{EMet} \arrow[u, bend right] \arrow[d, Rightarrow] \\
T_{\mathscr{U}} \quad \otimes & T_{\mathscr{U}'} & \overset{?}{\cong} & T_{\mathscr{U}+\mathscr{U}'}
\end{tikzcd}
$$

Tensor product of $\mathscr{U} + \mathscr{U}'$

Tensor product of monads

Open Problem 4

Quantitative theory of effects

(contribute to probabilistic programming languages)

Thank you
for the attention