Converging from Branching to Linear Metrics on MCs

(theoretical aspects)

Giorgio Bacci, Giovanni Bacci, Kim G. Larsen, Radu Mardare
Aalborg University, Denmark

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IDEA4CPS

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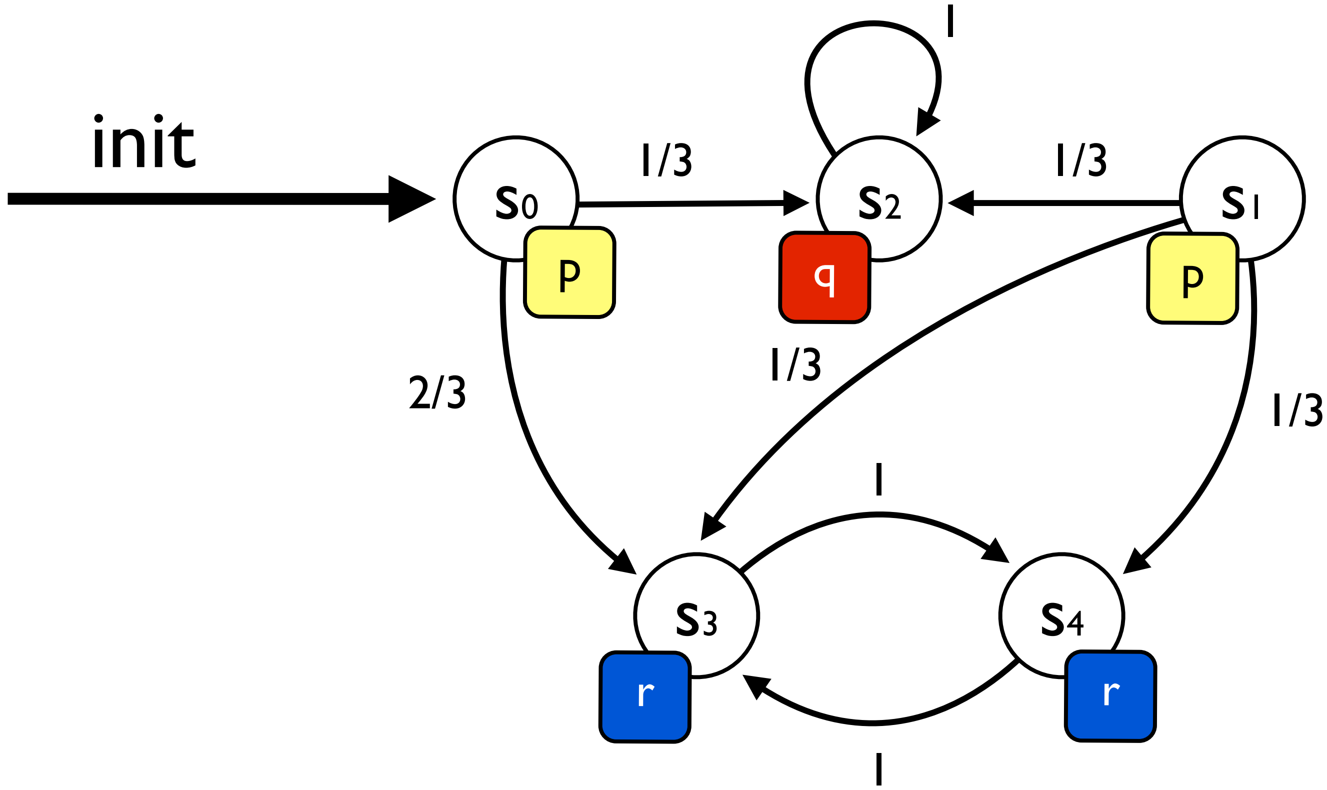
  - **Why?** --systems biology, machine learning, artificial intelligence, security, etc.

Markov Chains



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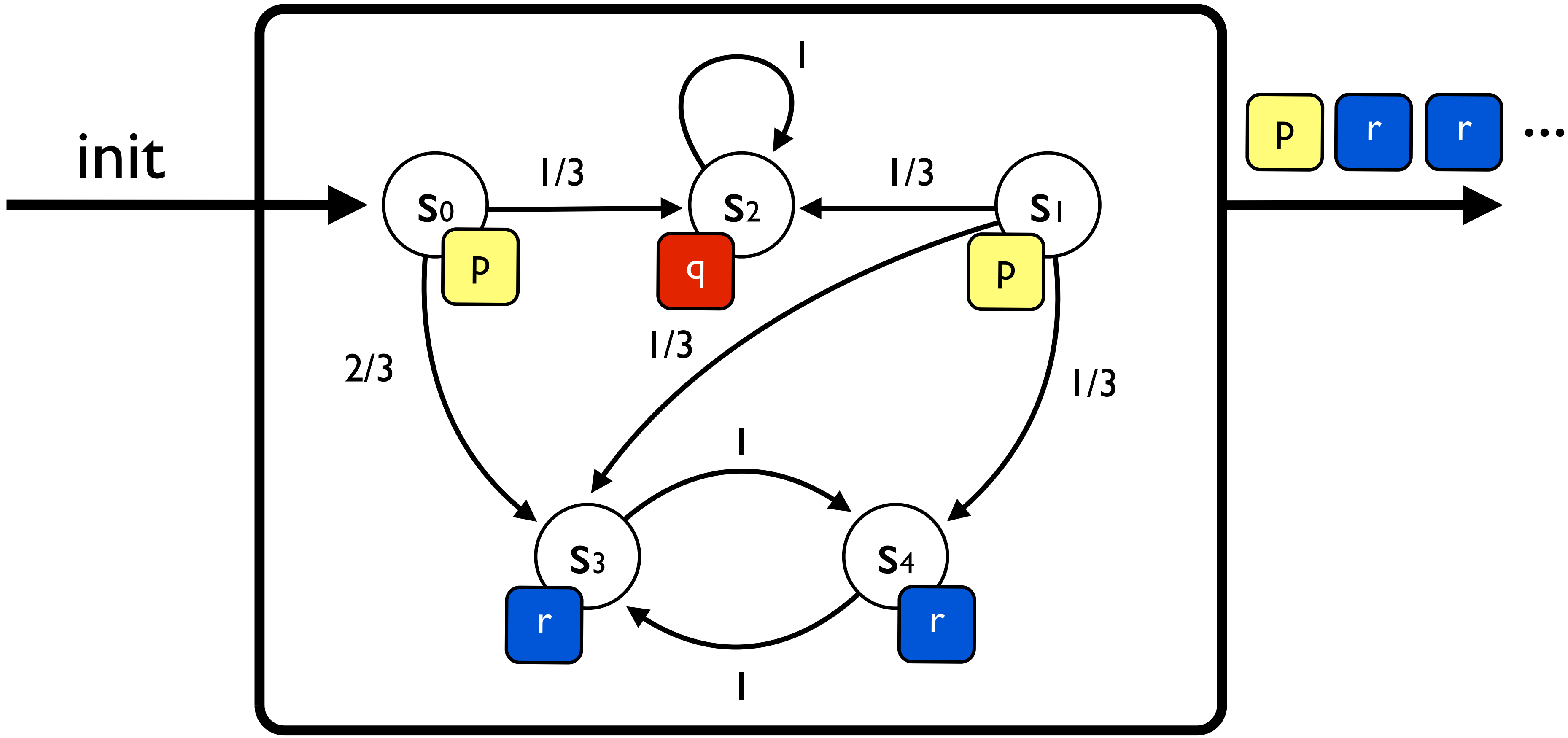
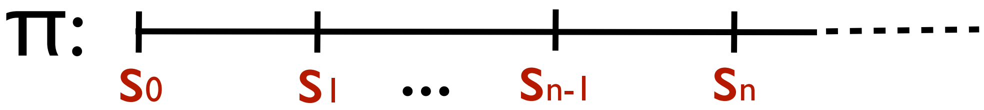


We are given "machines" that emit infinite traces of symbols with a certain probability

Measurable Events



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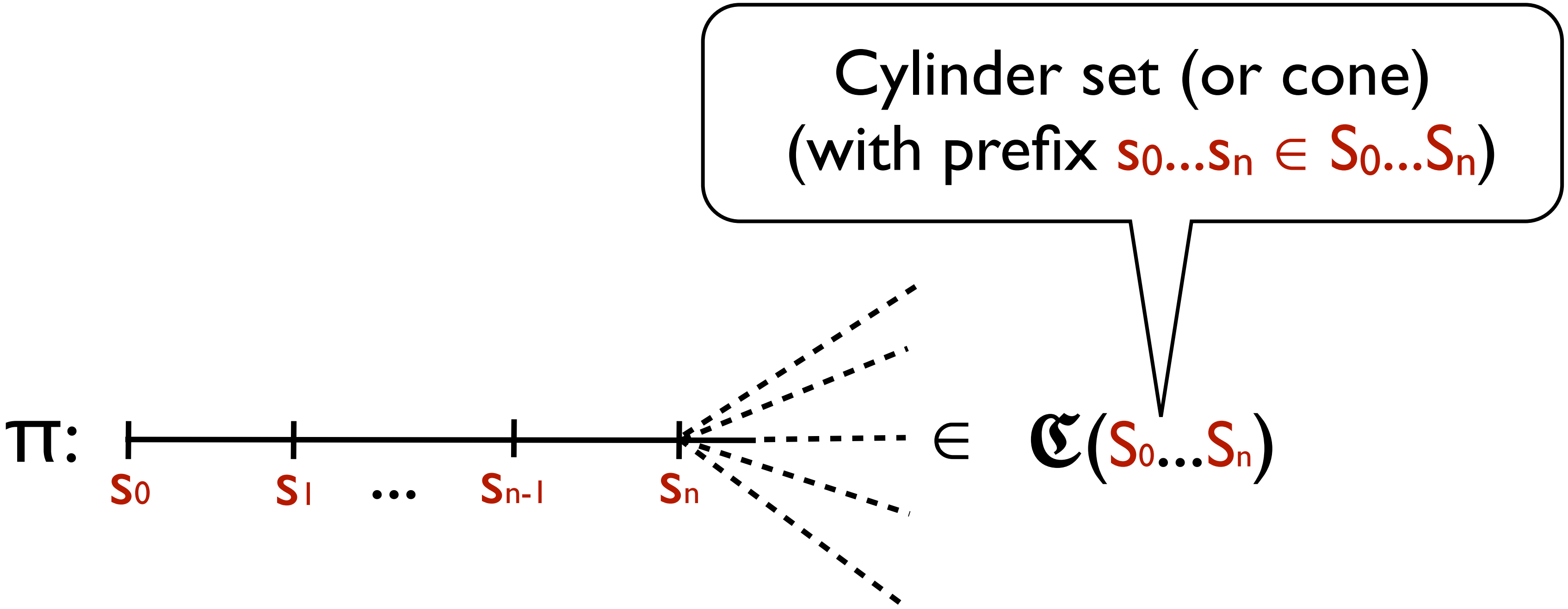
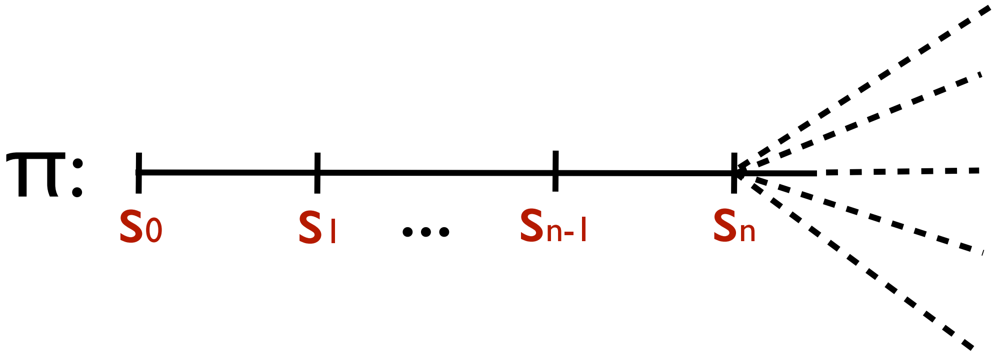


Measurable Events

Cylinder set (or cone)
(with prefix $s_0 \ldots s_n \in S_0 \ldots S_n$)



$\in \ \mathfrak{C}(S_0 \ldots S_n)$

$P(s)(\mathfrak{C}(S_0 \ldots S_n)) =$ "probability that, *starting from s*, the MC emits a path with prefix in $S_0 \ldots S_n$"

Measurable Events

Cylinder-set construction

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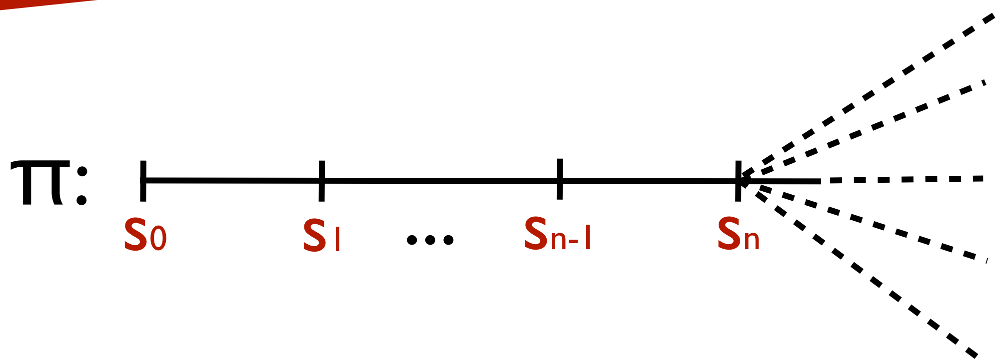

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(Pnueli)

Linear Temporal Logic

Atomic prop.

Next

Until

$$\varphi \coloneqq p \mid \bot \mid \varphi \rightarrow \varphi \mid X\varphi \mid \varphi U \varphi$$

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Semantics of a formula

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(Pnueli)

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Semantics of a formula

$$[\varphi] = \{\pi \mid \pi \underbrace{\models}_{\text{with usual satisfiability relation}} \varphi\}$$

Probabilistic
Model Checking

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$$P(s)([\varphi]) = ?$$

What is the probabability that the MC with initial state s satisfies the formula φ?

Approximate verification

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- **Proposed solution:**
  Behavioral metrics for quatifying the error

A distance for model checking

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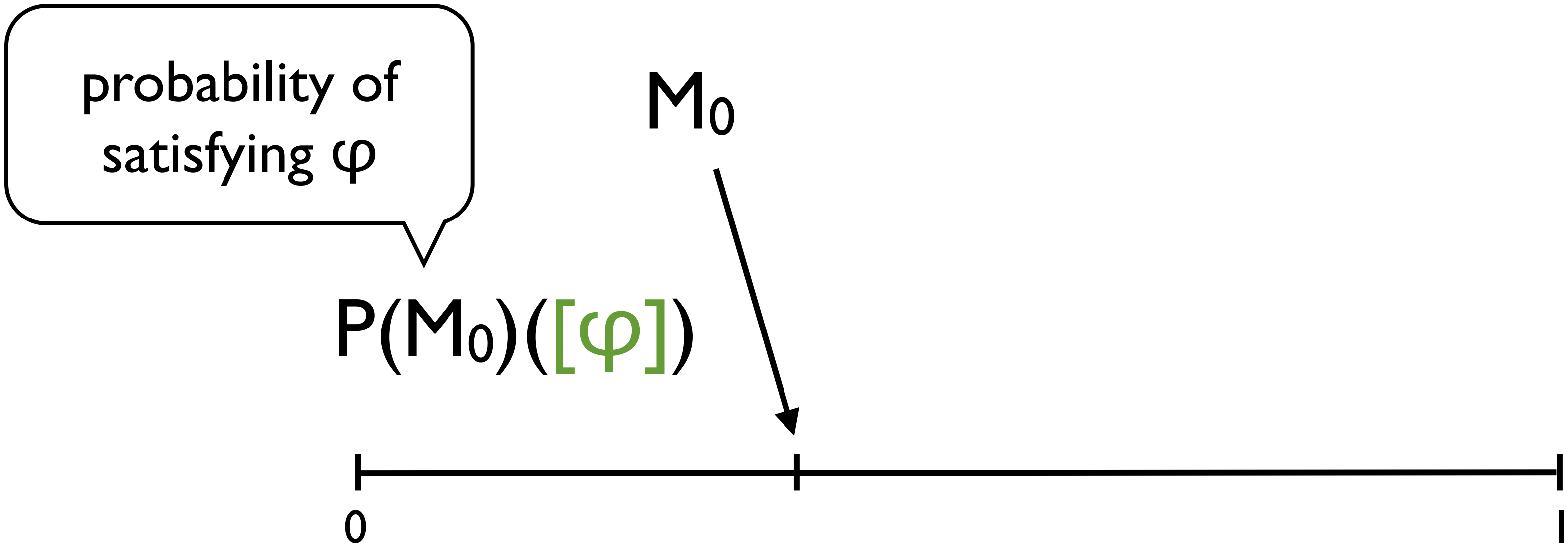

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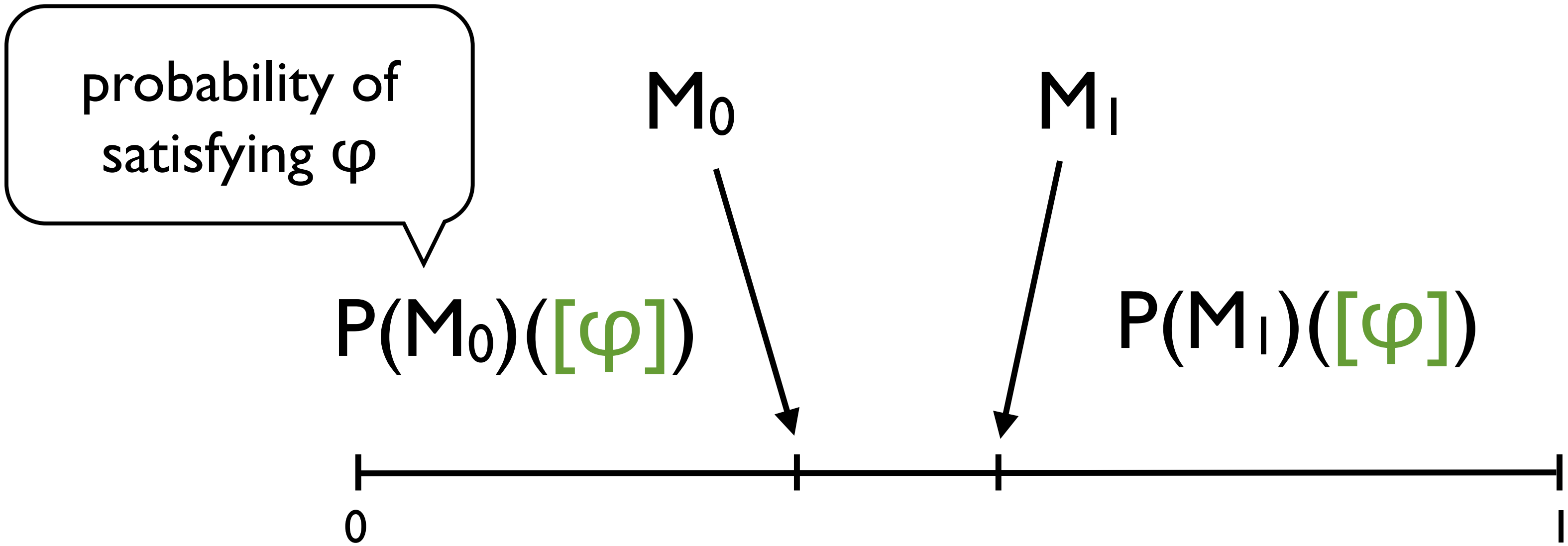

$$|P(M_0)([\varphi]) - P(M_1)([\varphi])|$$

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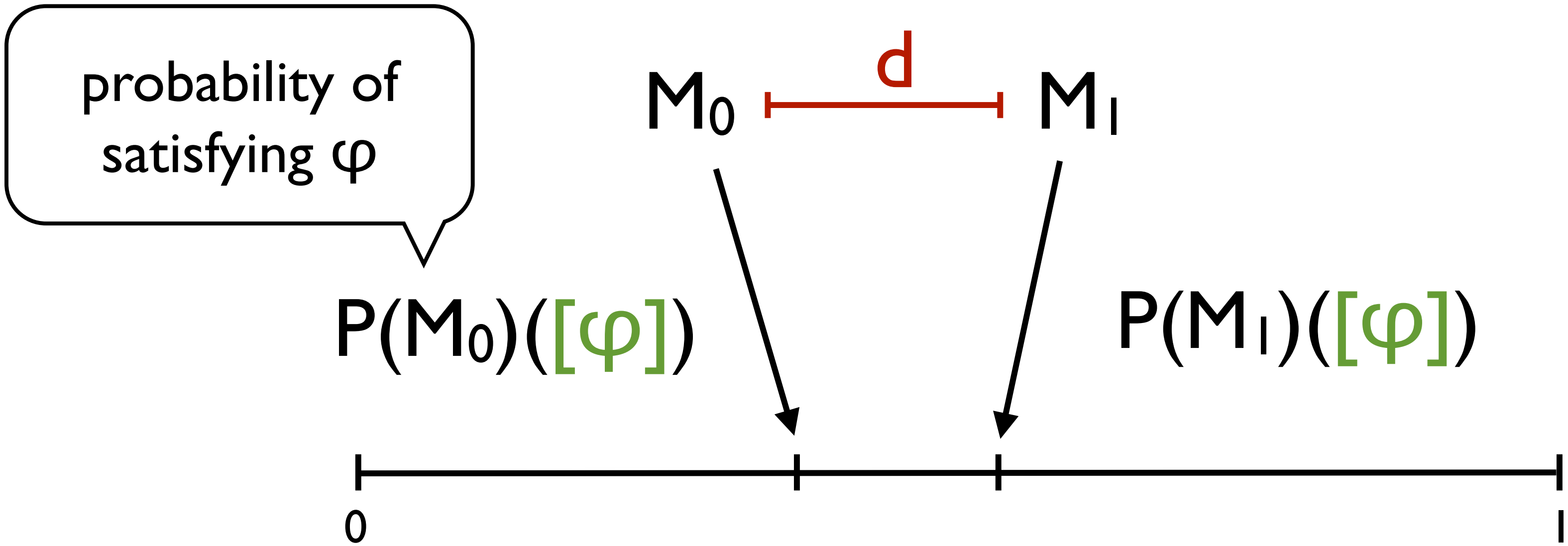

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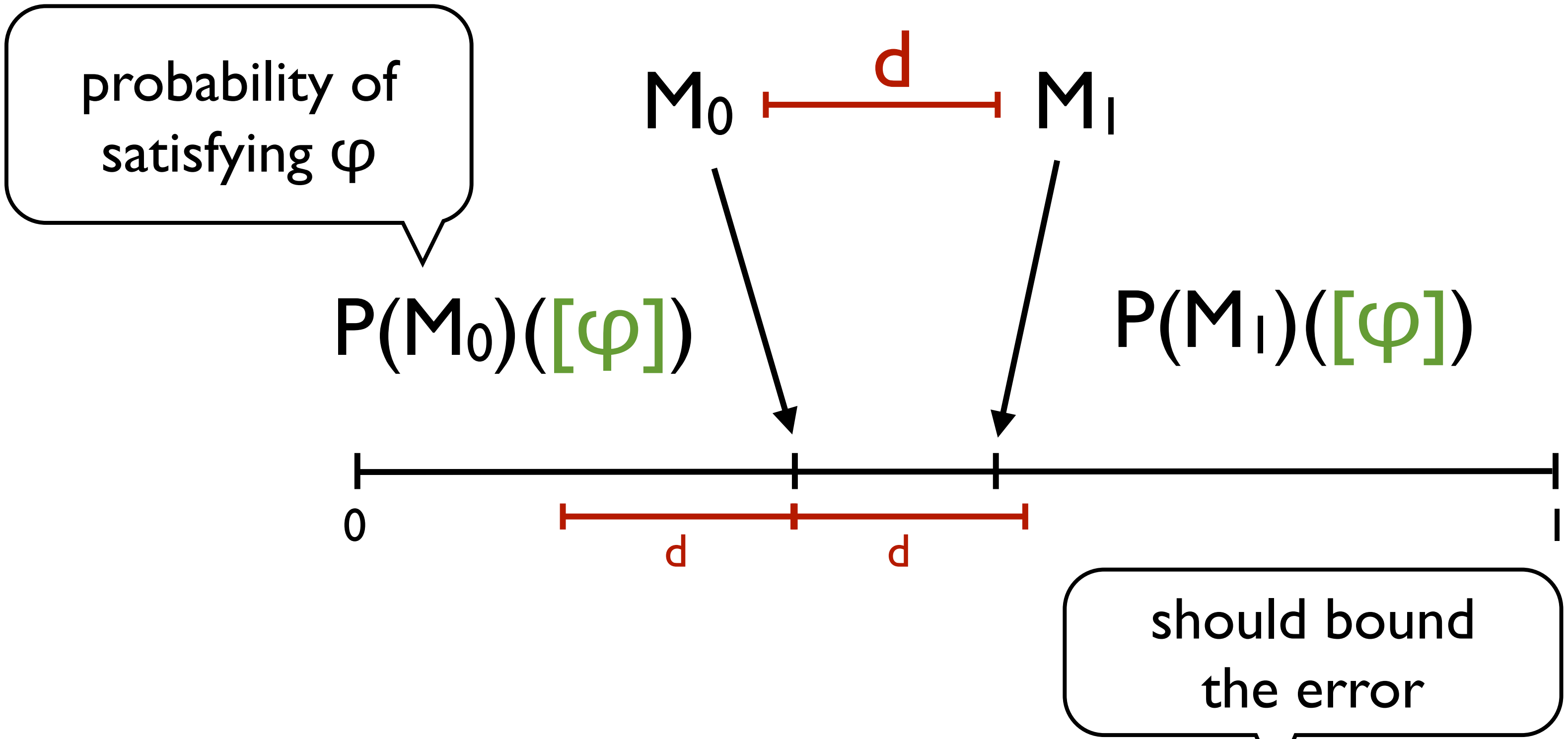

$$|P(M_0)([\varphi]) - P(M_1)([\varphi])| \leq d$$

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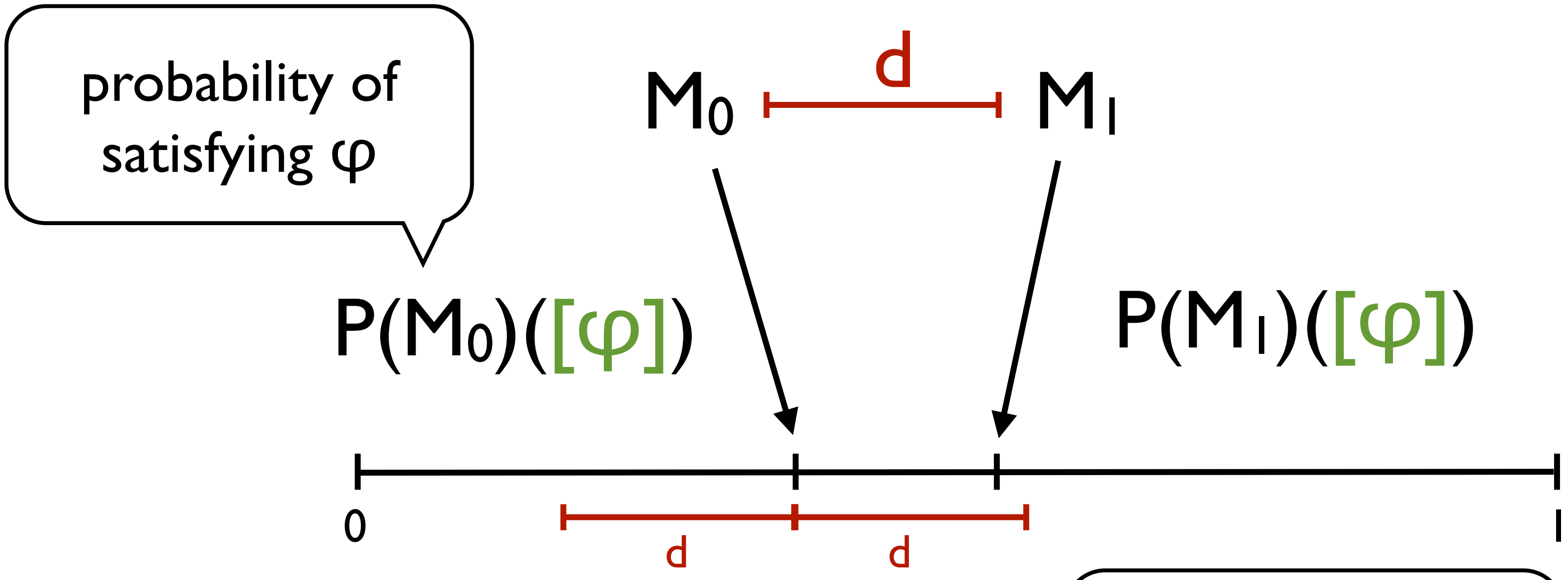

should bound the error

$$|P(M_0)([\varphi]) - P(M_1)([\varphi])| \leq d$$

for all formulas!

Two logical distances

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the LTL distance

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LTL without next operator

Two logical distances

Three natural questions

Q1: Can we compute the two metrics?

Q2: Can we compute them exactly?
If not, can we approximate them
to any arbitrary precision?

Q3: What about complexity?

Characterizations

Trace distance

$$T(s,t) = \sup_{E \in \sigma(\mathcal{T})} |P(s)(E) - P(t)(E)|$$

Stutter-trace distance

$$ST(s,t) = \sup_{E \in \sigma(\mathcal{ST})} |P(s)(E) - P(t)(E)|$$

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Events up-to trace equivalence

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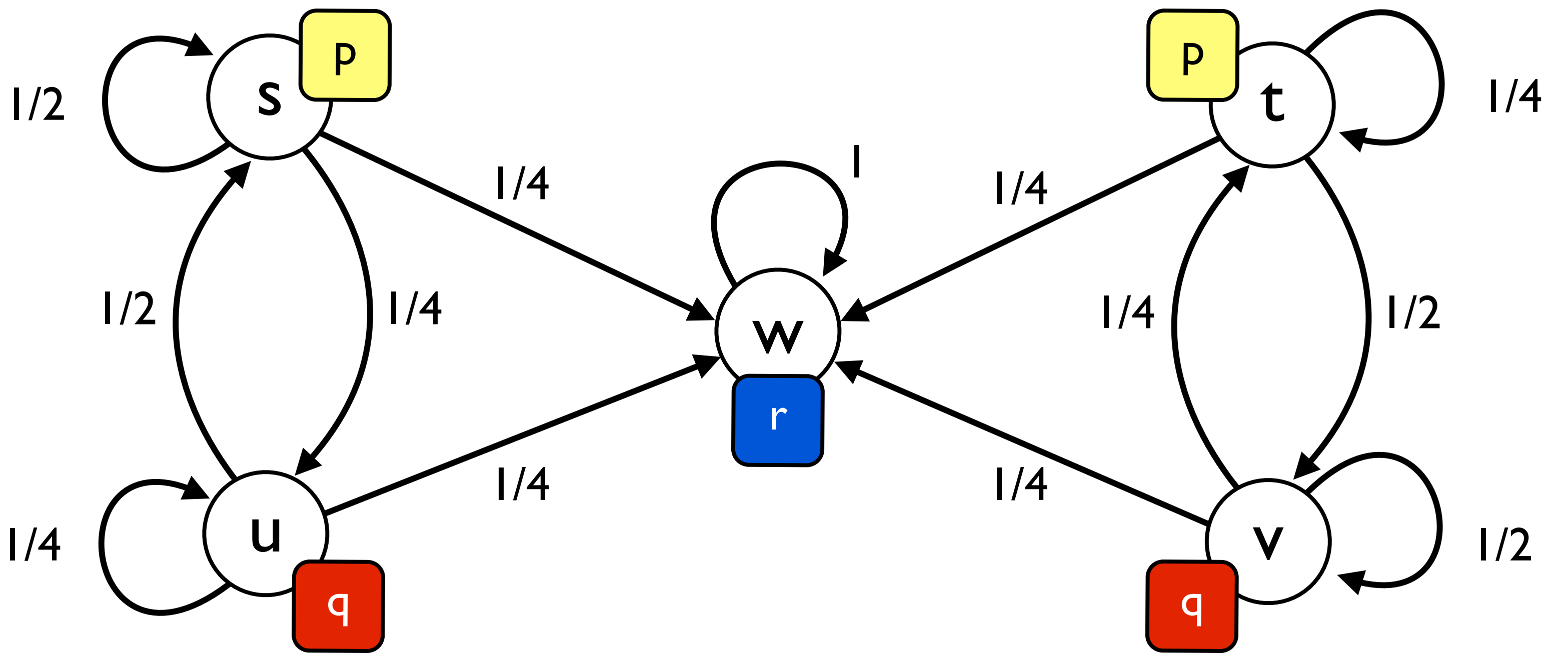
Characterization Theorem

$LTL(s,t) = T(s,t)$ and $LTL^{-x}(s,t) = ST(s,t)$

A tiny yet tricky example

(from Chen-Kiefer LICS'14)



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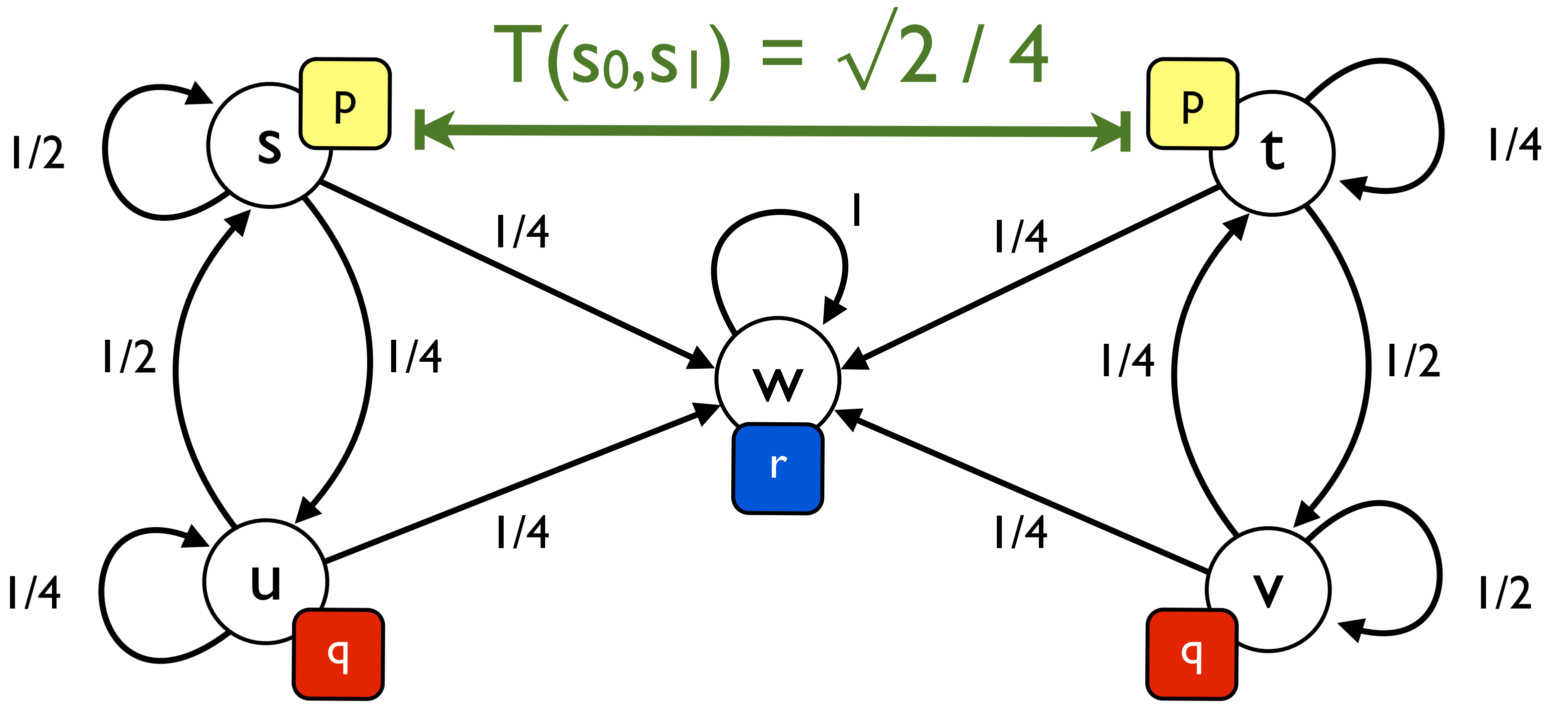

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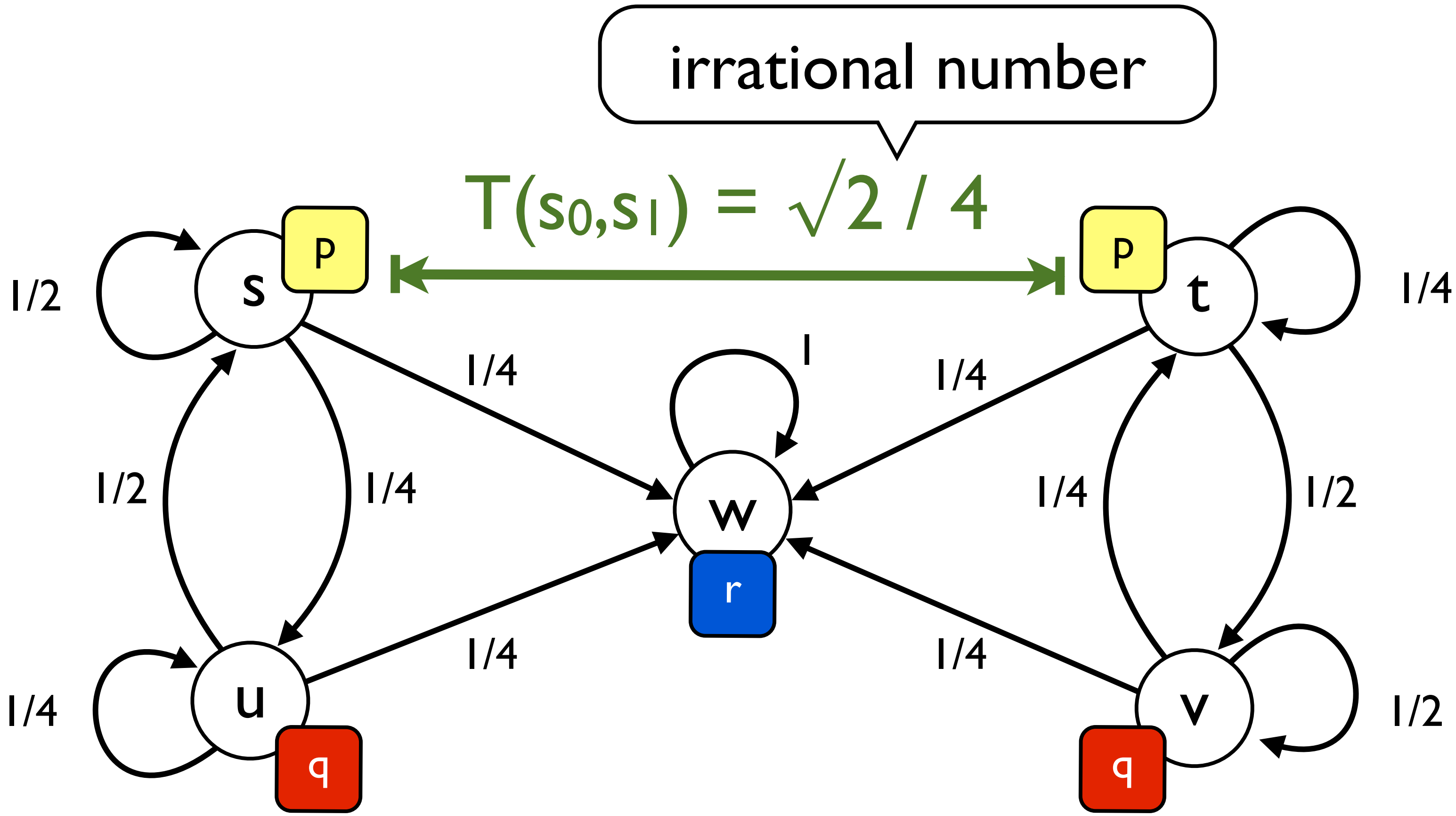

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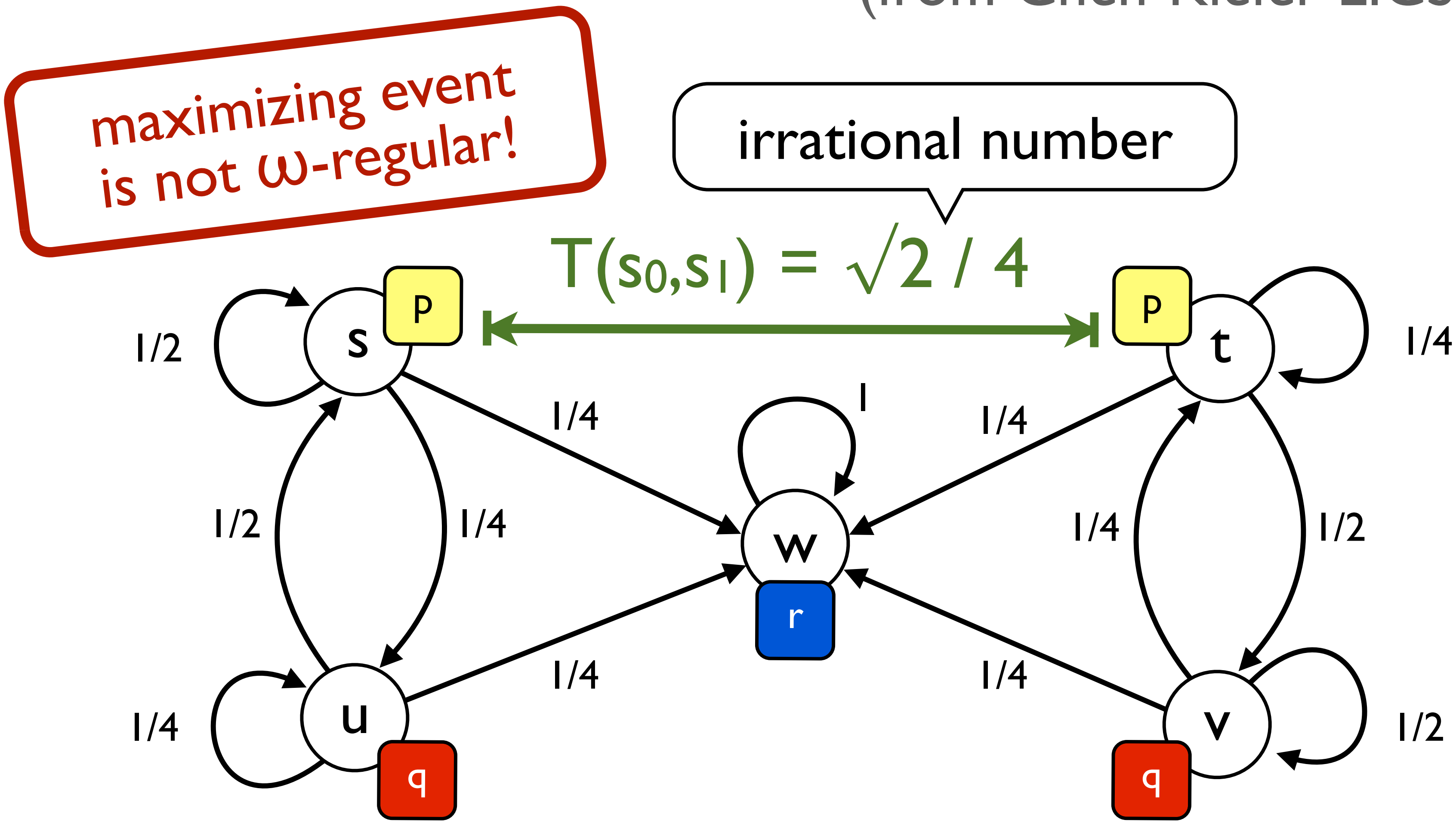

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Q: Can we approximate the logical/trace distances up to any arbitrary precision?

Approximation Algorithm

(in the slides only for the Trace Distance)

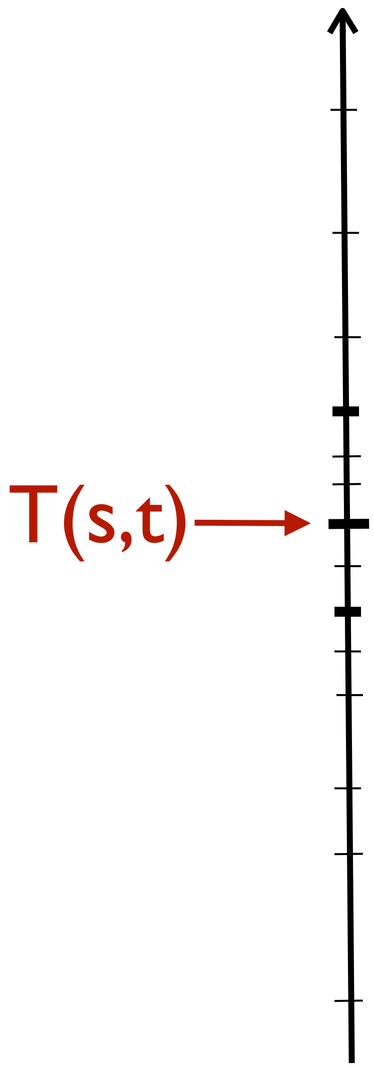
generalizes / improves
Chen-Kiefer LICS'14

Approximation Algorithm

(in the slides only for the Trace Distance)

(general idea)

Approximation Schema



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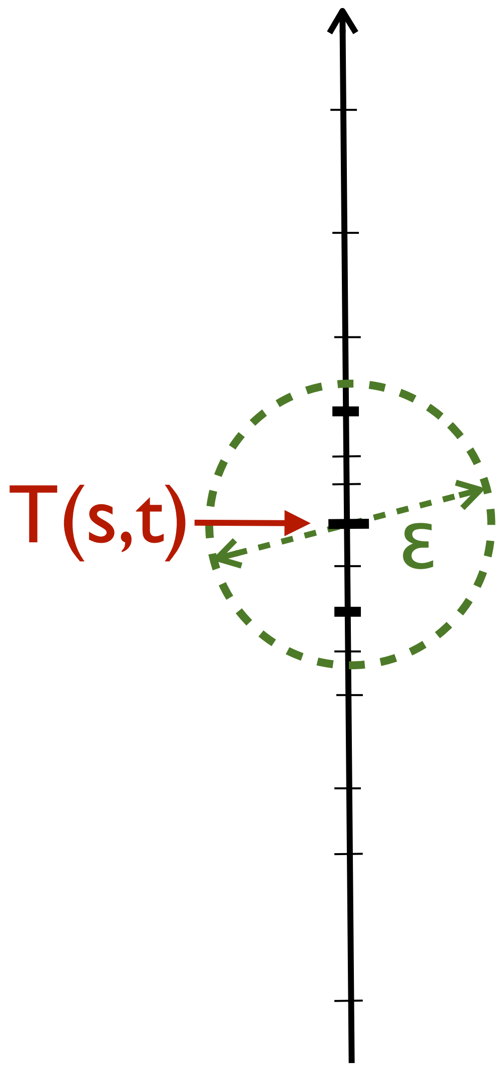

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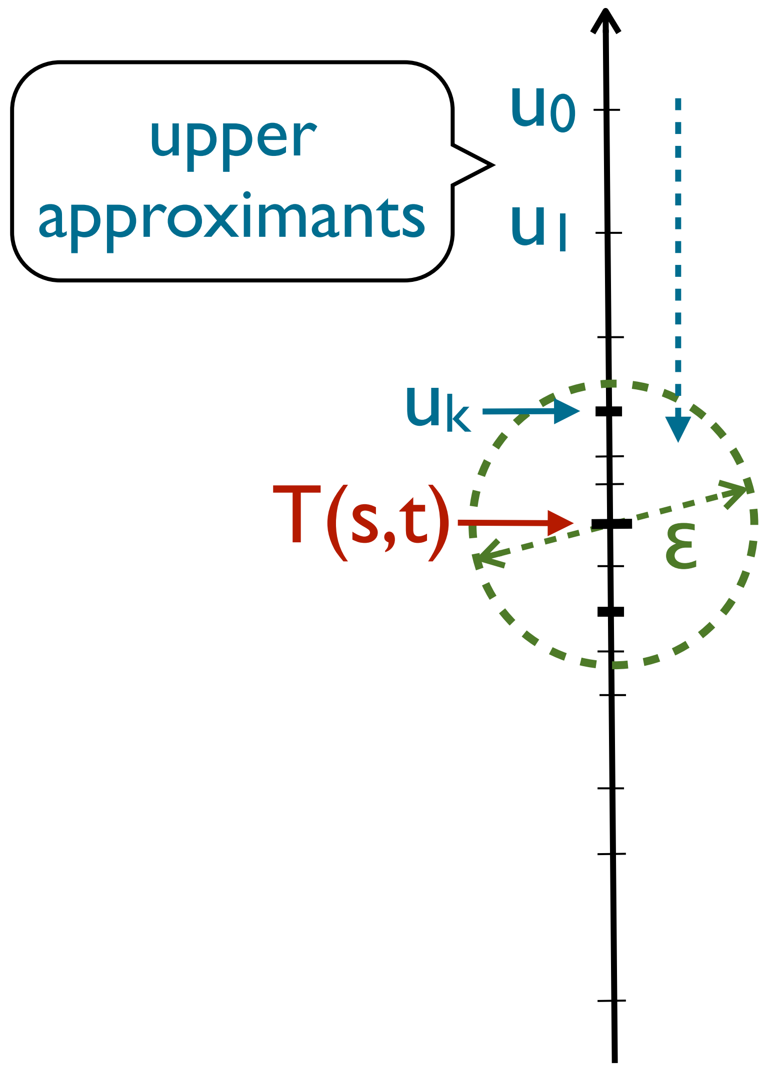

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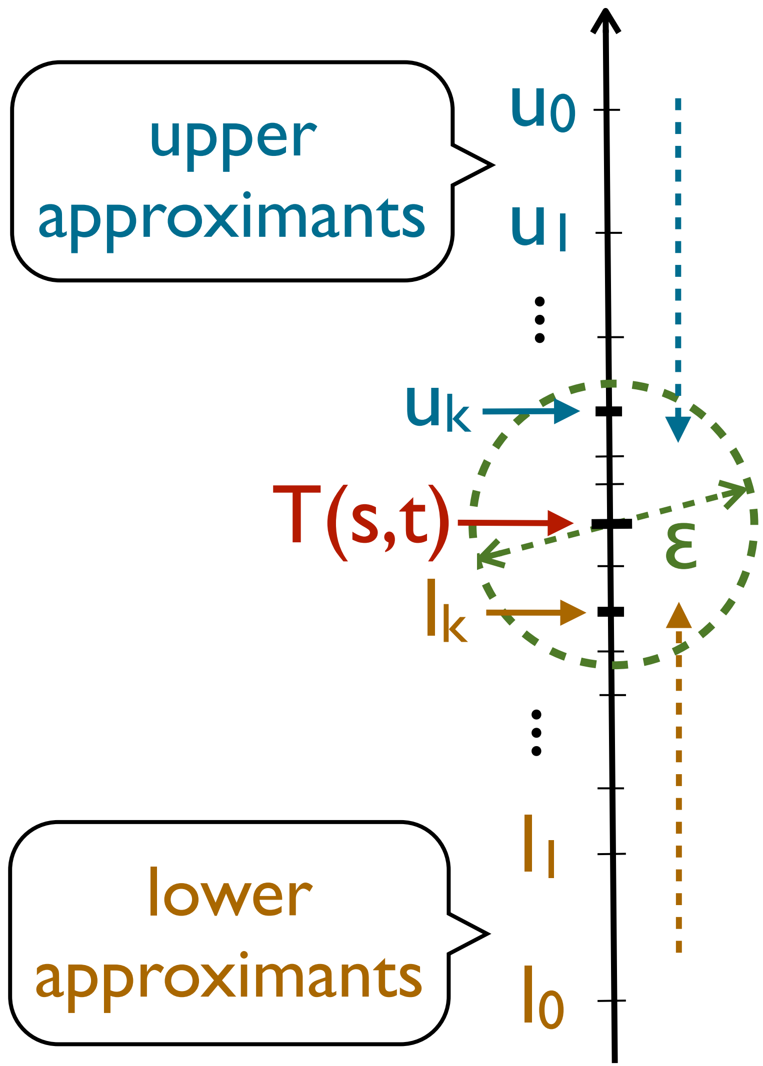

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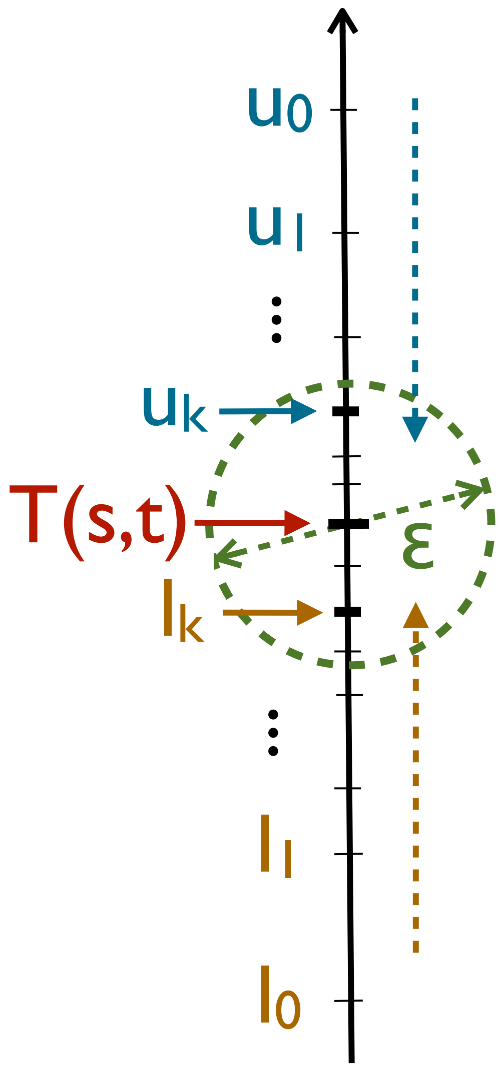

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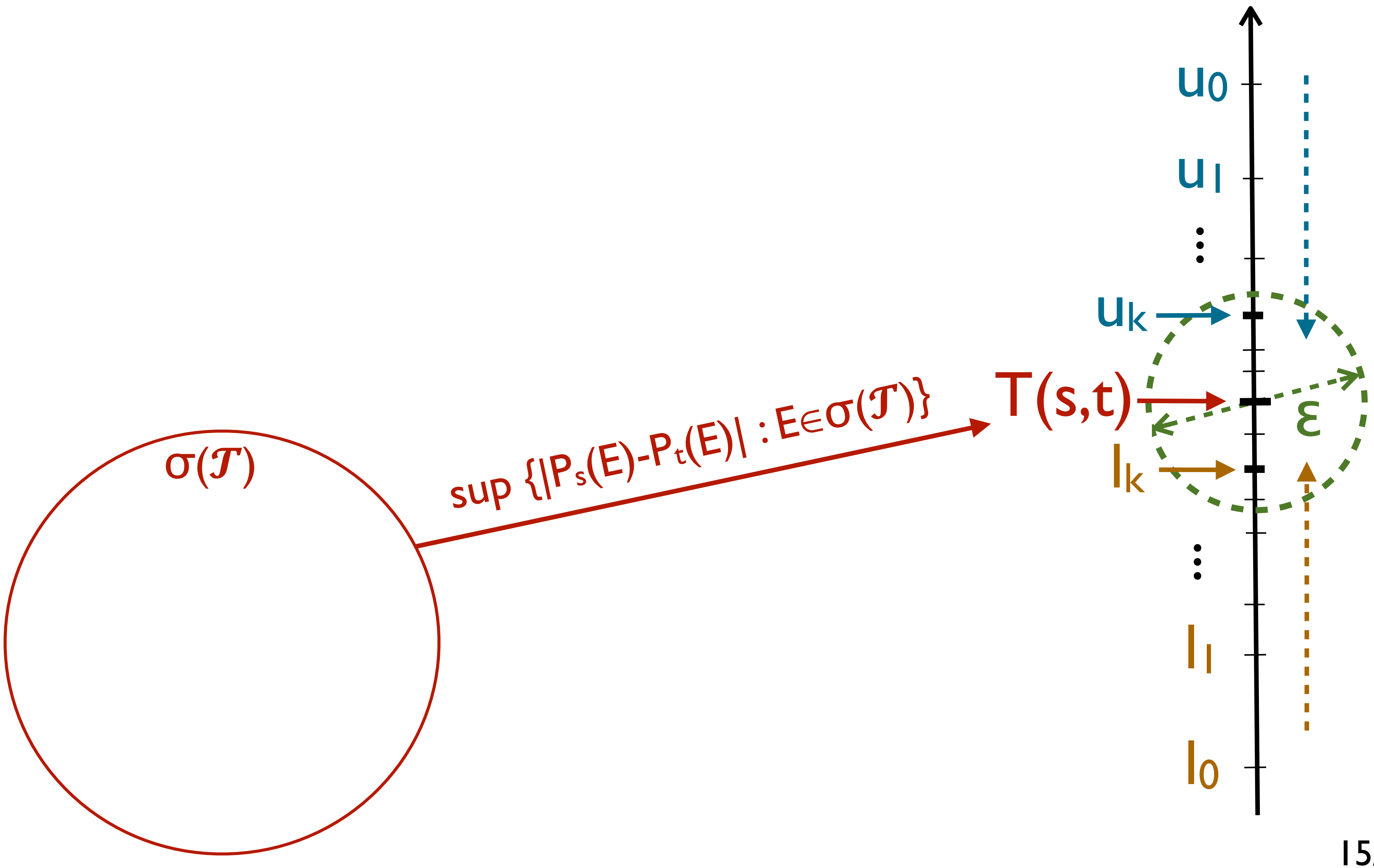

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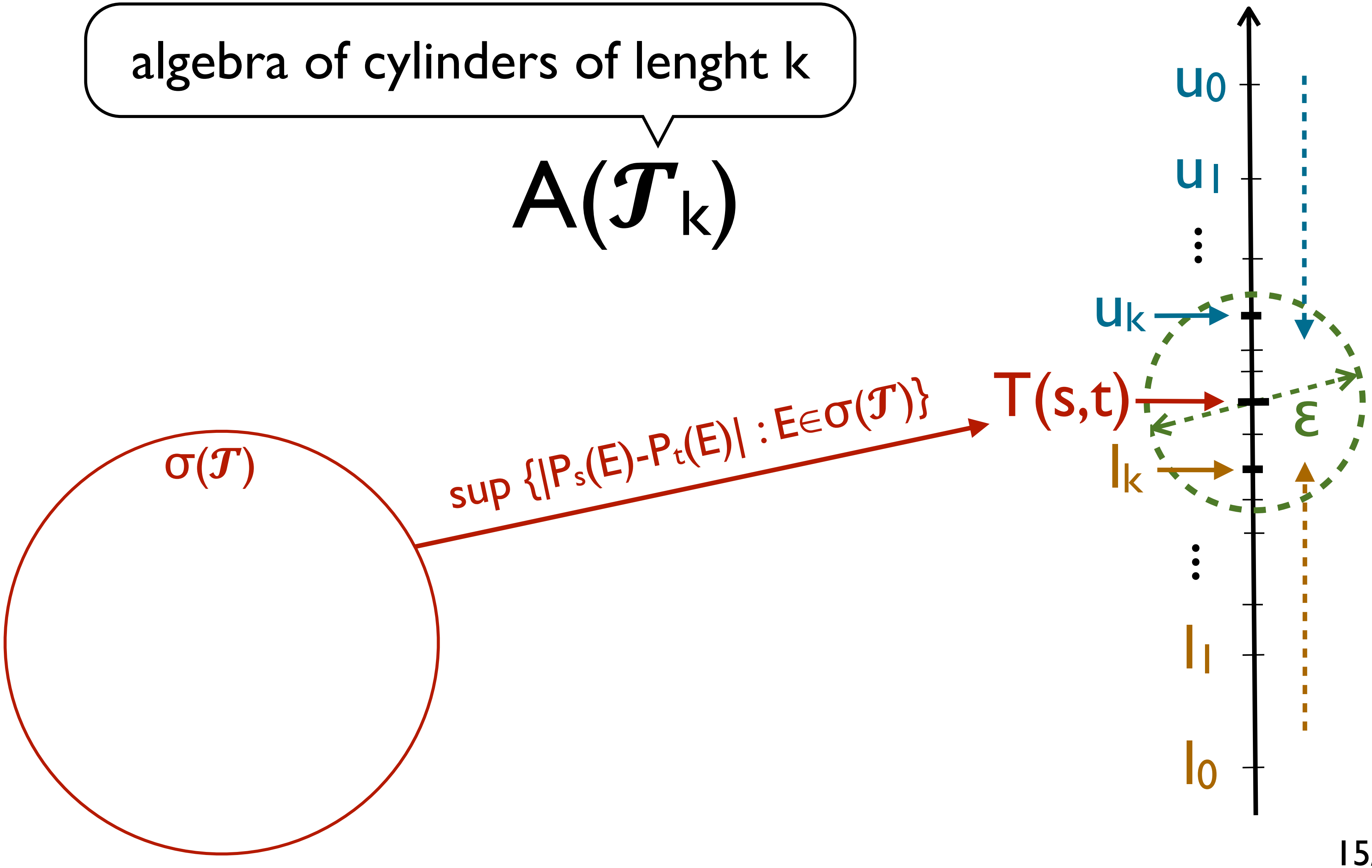

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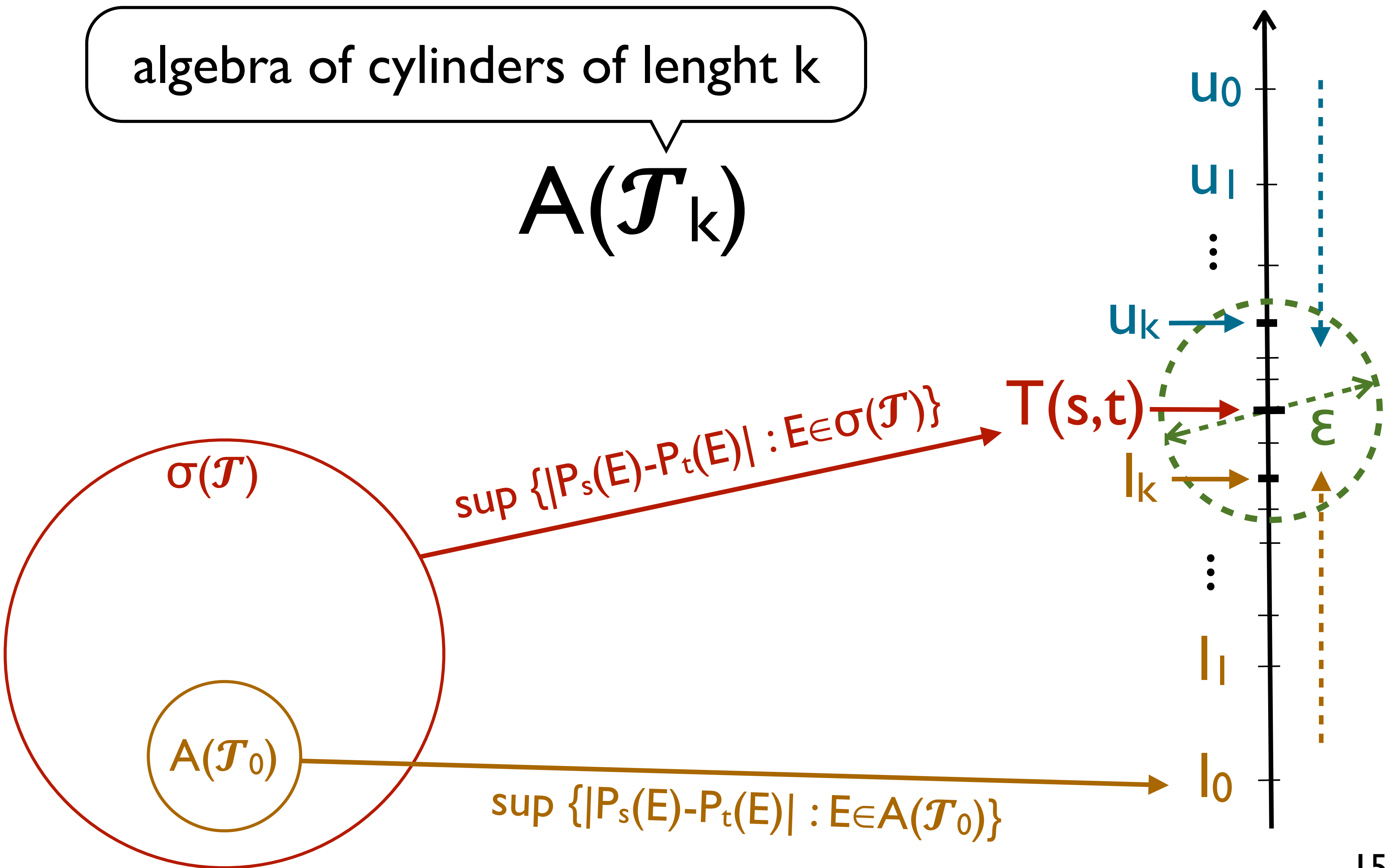

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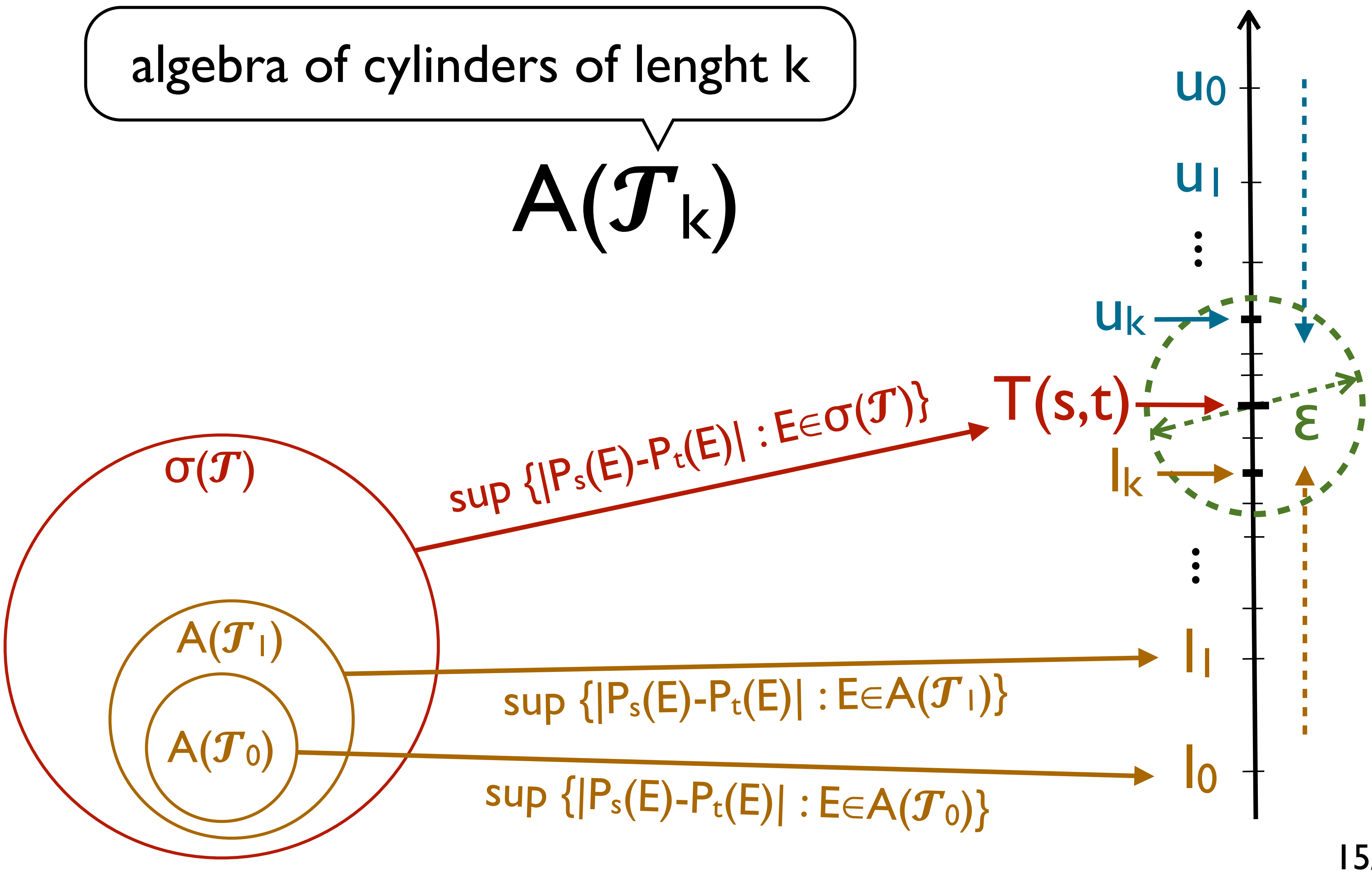

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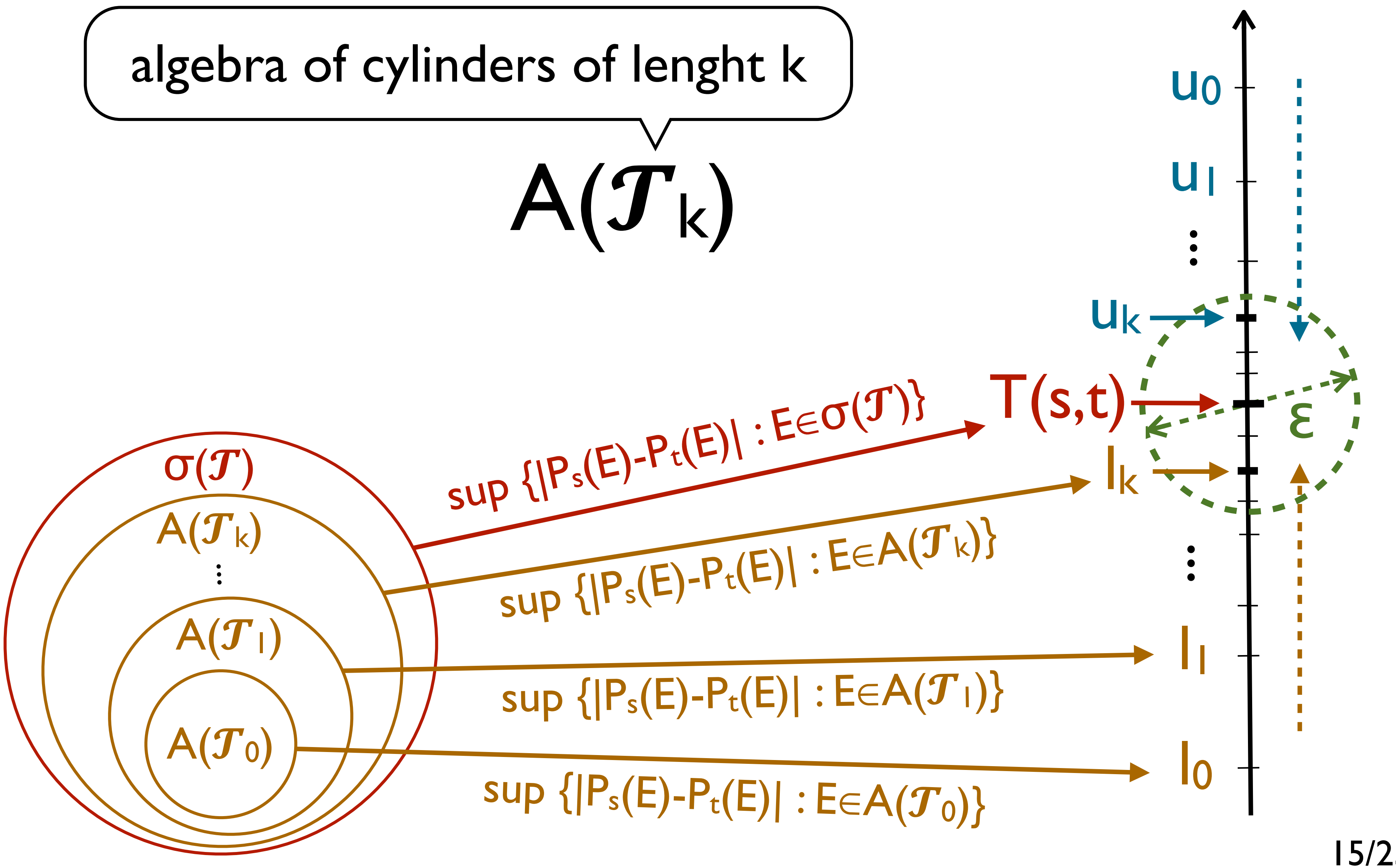
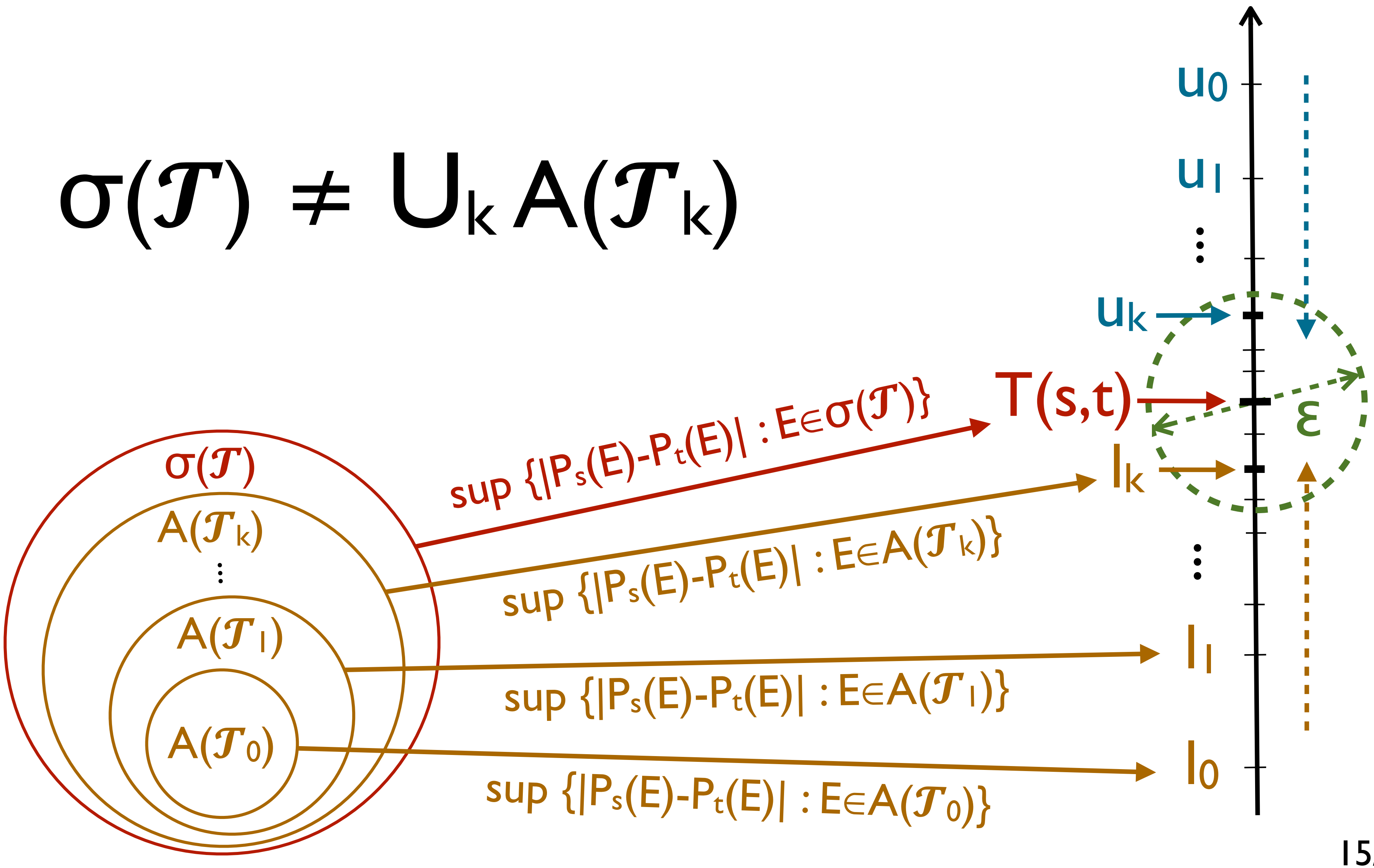


Approximation Schema (general idea)

$$\sigma(\mathcal{T}) \neq \bigcup_k A(\mathcal{T}_k)$$



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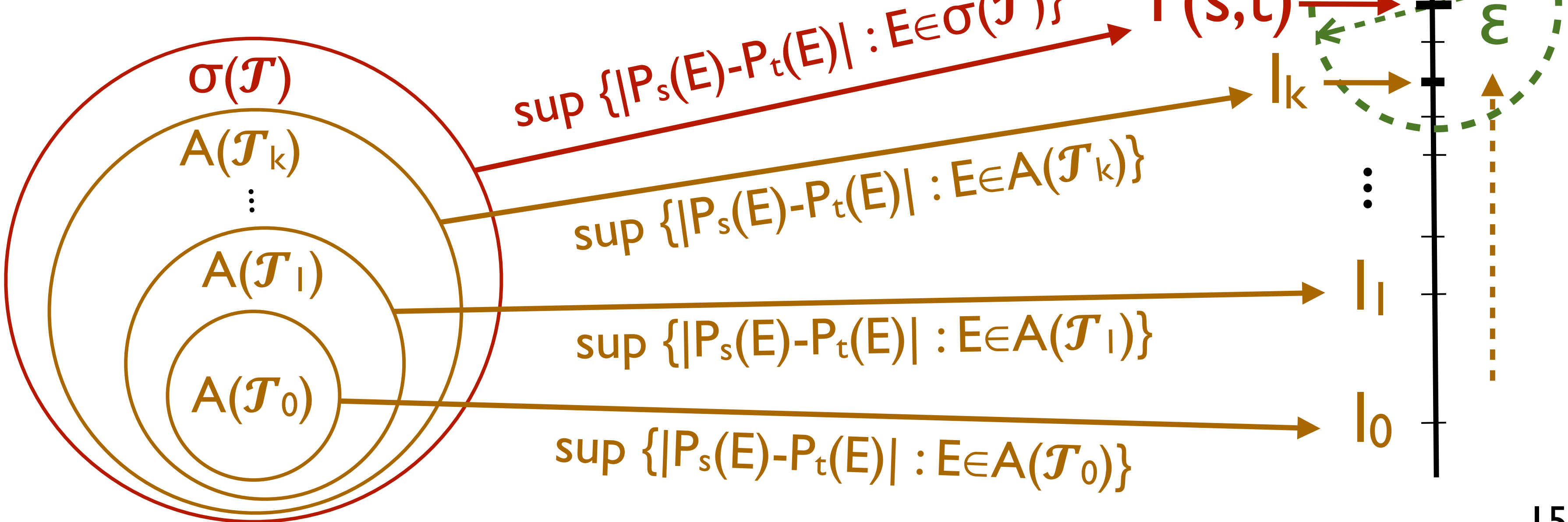
dense subset



Coupling Characterization
(as total variation distance)

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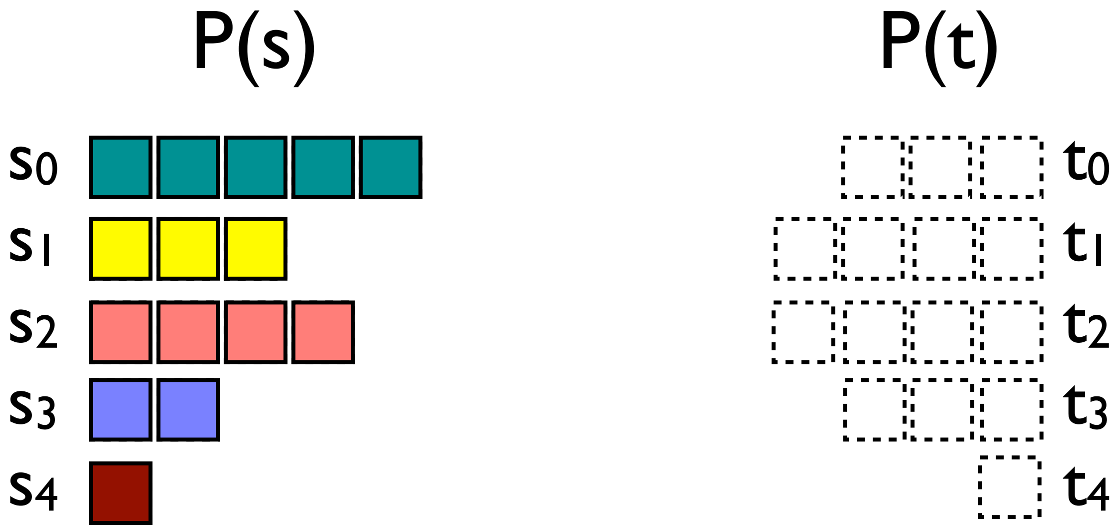
Coupling as a transportation schedule...



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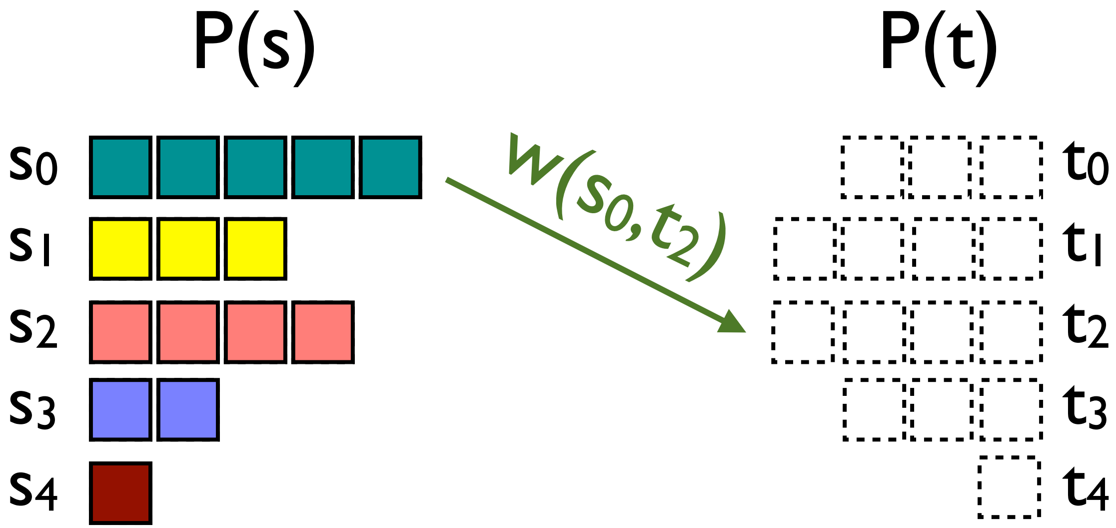

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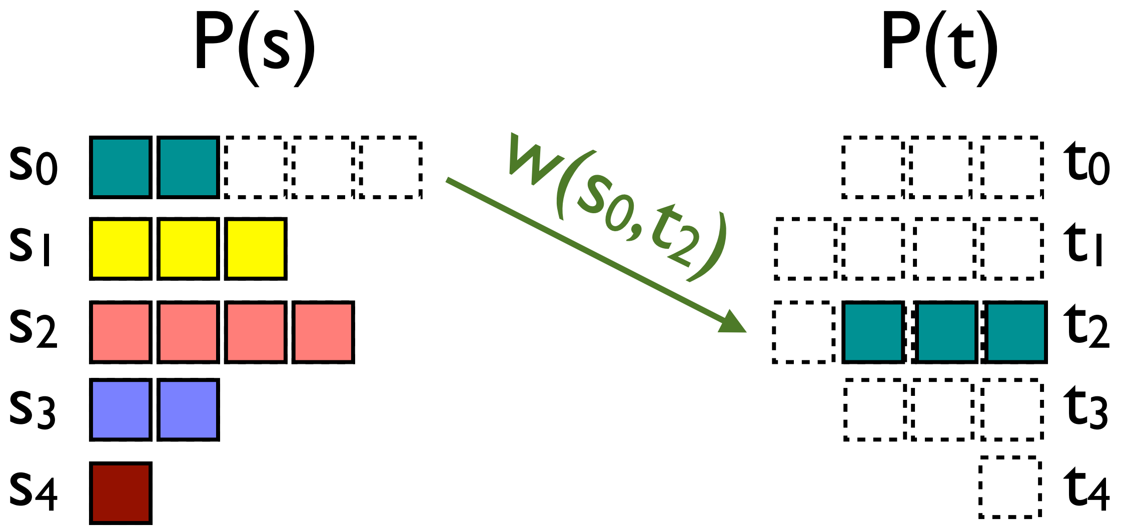

Coupling Characterization

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trace inequivalence

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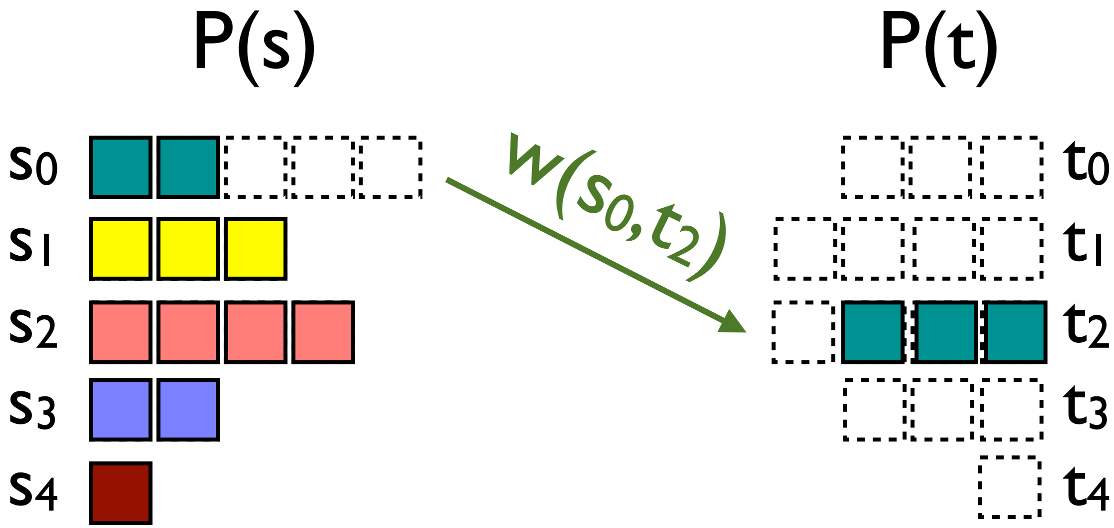

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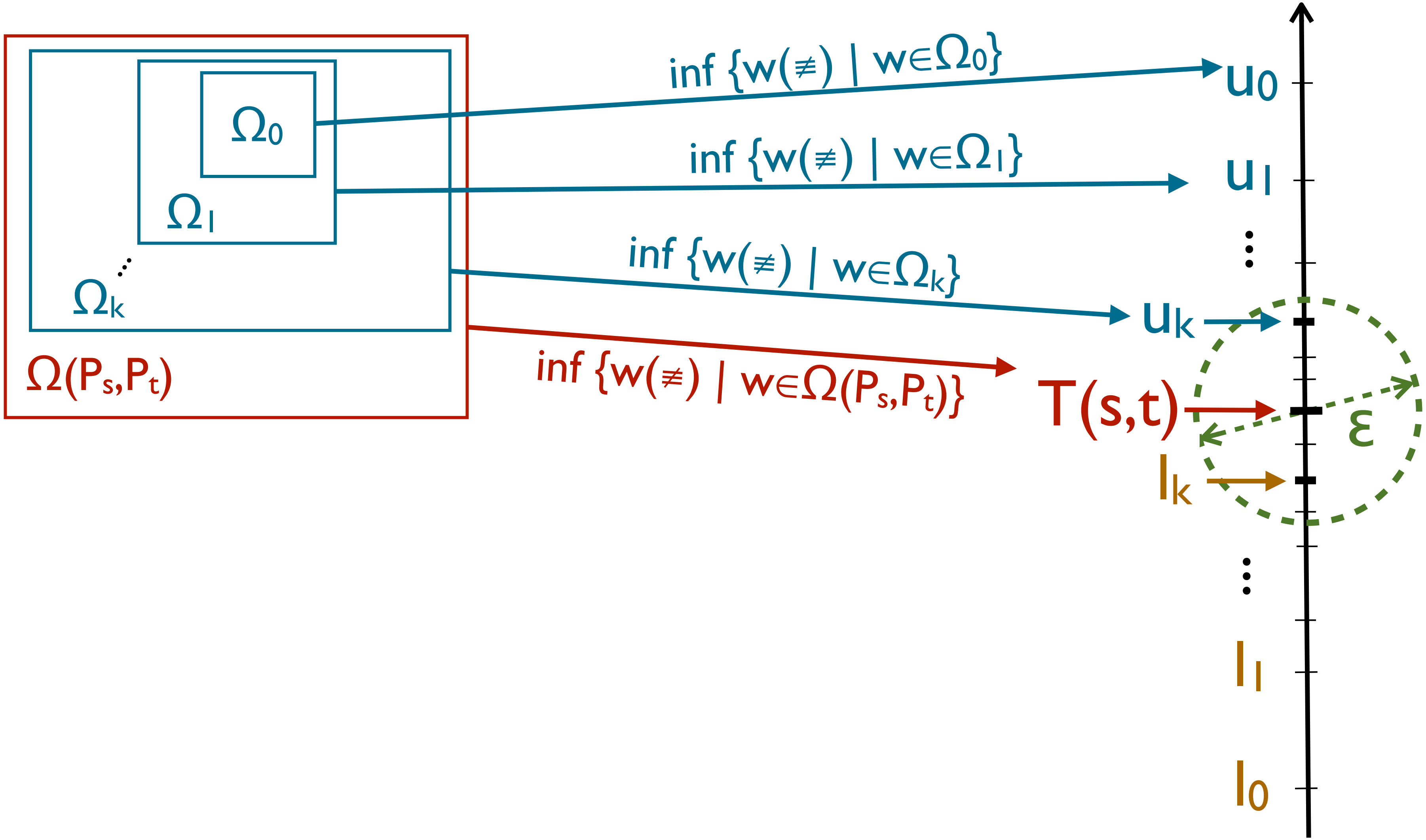

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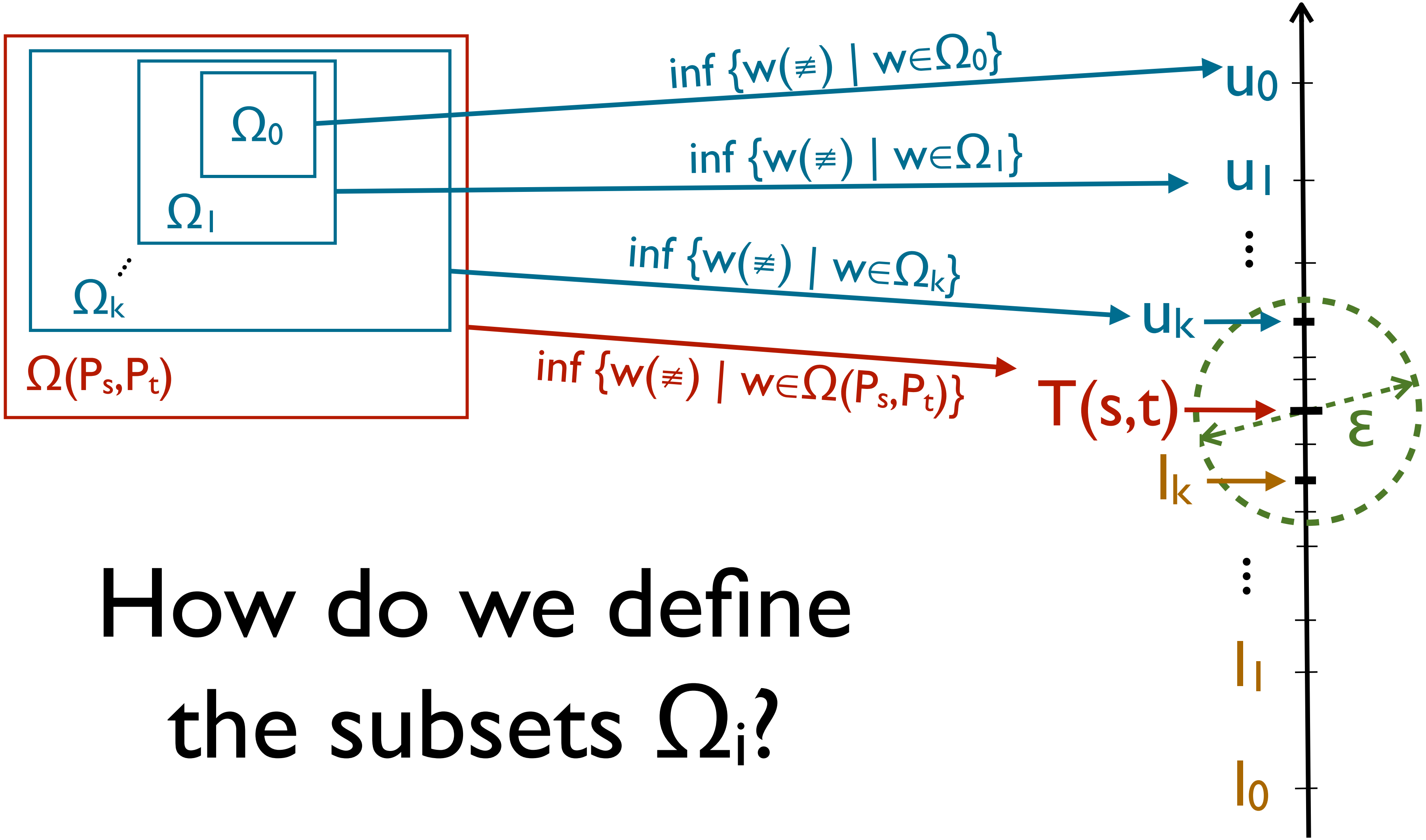
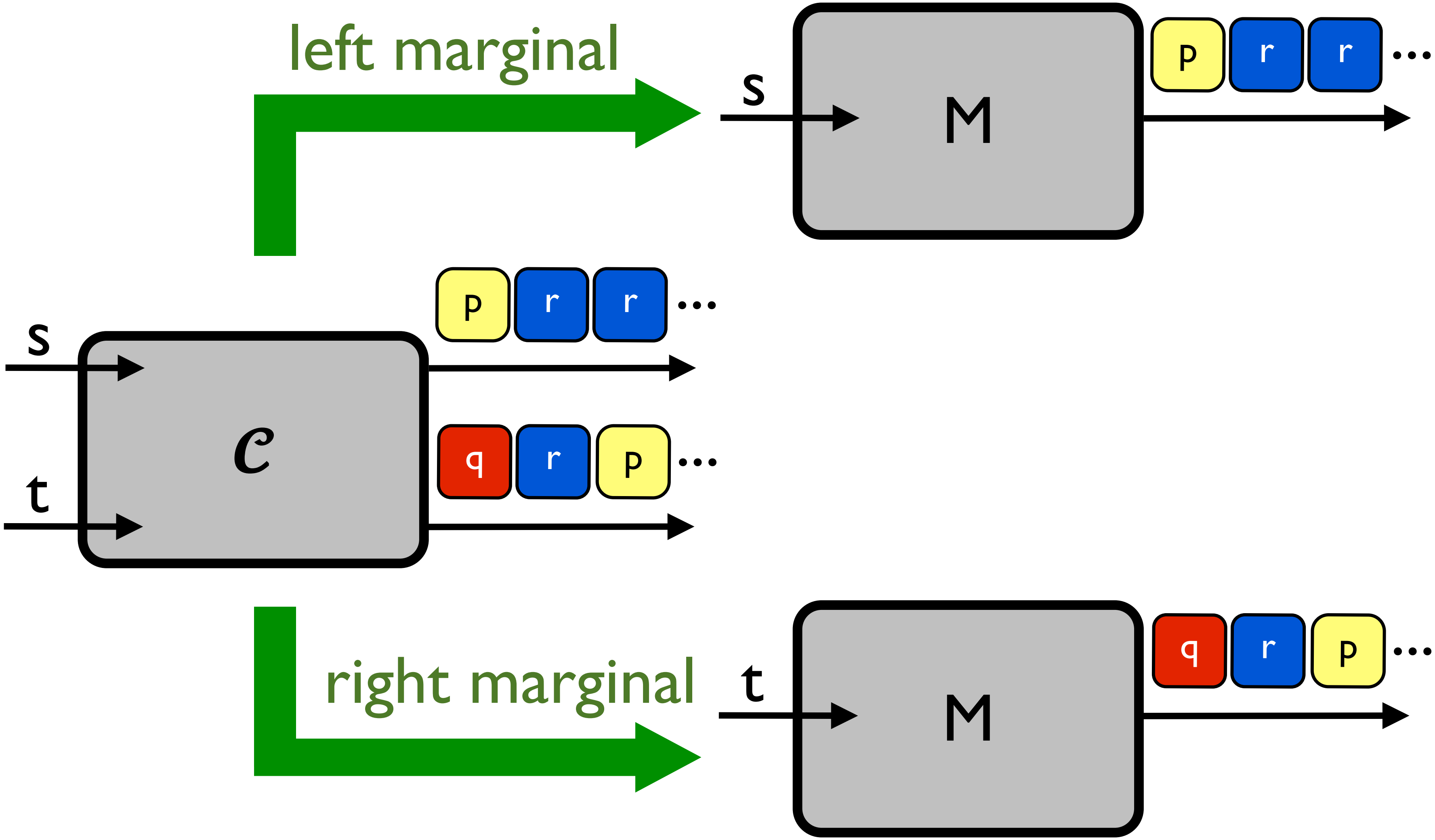


How do we define the subsets Ω_i?

Coupling Structures



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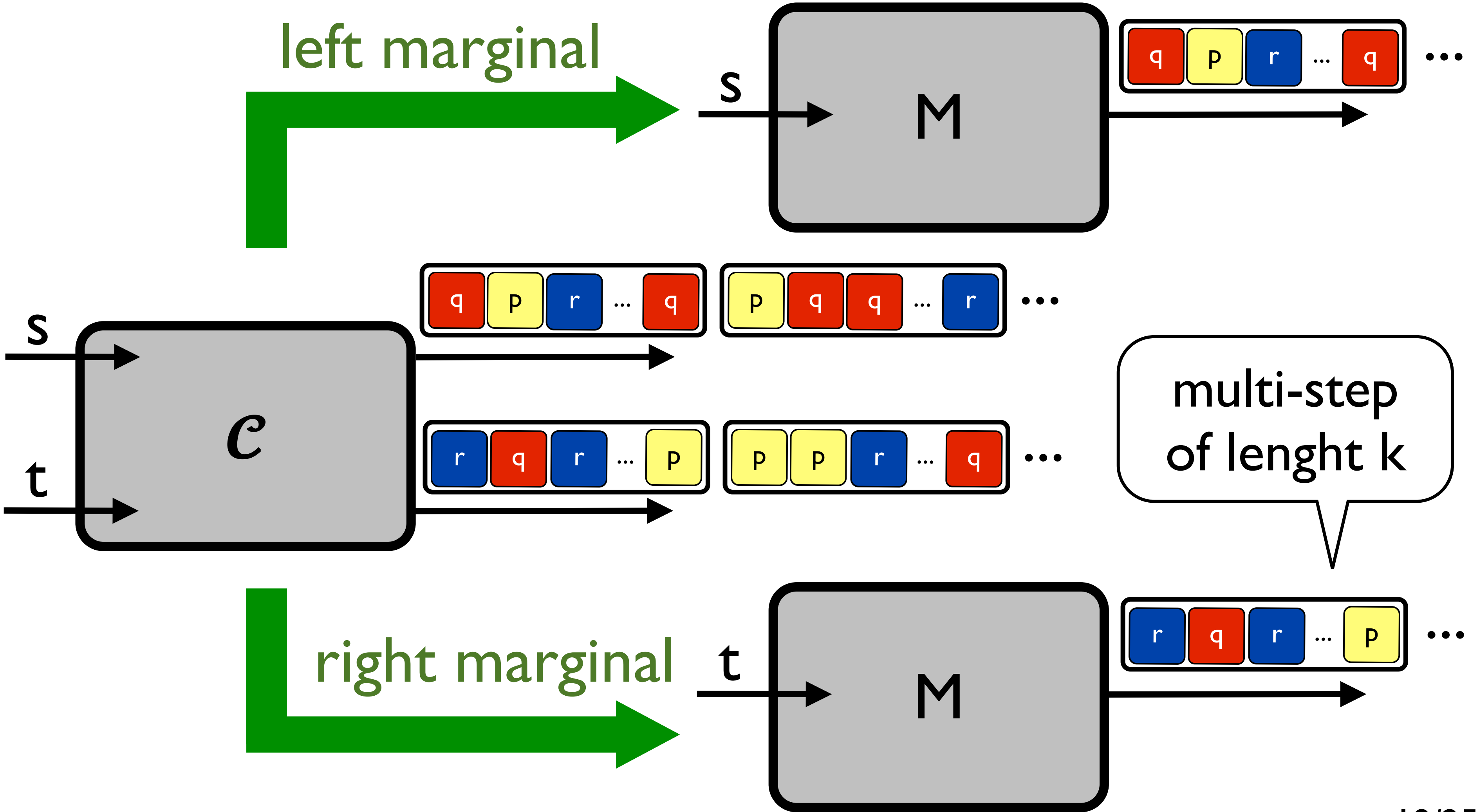

Coupling Structure of rank k

$$\mathcal{C}: S \times S \to \Delta(S^k \times S^k)$$

such that $\mathcal{C}(s,t) \in \Omega(P(s)^k, P(t)^k)$

the model in the box

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Probability induced by $\mathcal{C}$ starting from (s,t)

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Lemma

(i) $\Omega_k \subseteq \Omega(P(s),P(t))$, (ii) $\Omega_k \subseteq \Omega_{hk}$ (for all k,h>0)

(iii) $\cup_k \Omega_k$ is dense in $\Omega(P(s),P(t))$

Upper approx. are Branching Metrics!

$$\Theta(d)(s,t) = \begin{cases} 1 & \text{if } s \neq t \\ K(d)(\tau(s),\tau(t)) & \text{otherwise} \end{cases}$$

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its kernel is Larsen-Skou probabilistic bisimilarity!

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$$\Theta^k(d)(s,t) = \begin{cases} 1 & \text{if } s \neq t \\ K(\Lambda^k(d))(\tau^k(s),\tau^k(t)) & \text{otherwise} \end{cases}$$

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Kantorovich lifting + $\neq$-selector

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Exact vs Metric semantics

k-approximant

trace distance

metrics: $u_1 \geq \ \ldots \ \geq u_k \geq \ \ldots \ \inf_k u_k = T$

equivalences: $\sim_1 \subseteq \ \ldots \ \subseteq \sim_k \subseteq \ \ldots \ \bigcup_k \sim_k$

probabilistic
k-bisimulation

Exact vs Metric semantics

k-approximant

trace distance

metrics: $u_1 \geq \ldots \geq u_k \geq \ldots \inf_k u_k = T$

equivalences: $\sim_1 \subseteq \ldots \subseteq \sim_k \subseteq \ldots \bigcup_k \sim_k \quad \approx$

probabilistic k-bisimulation

probabilistic trace equiv.

Exact vs Metric semantics

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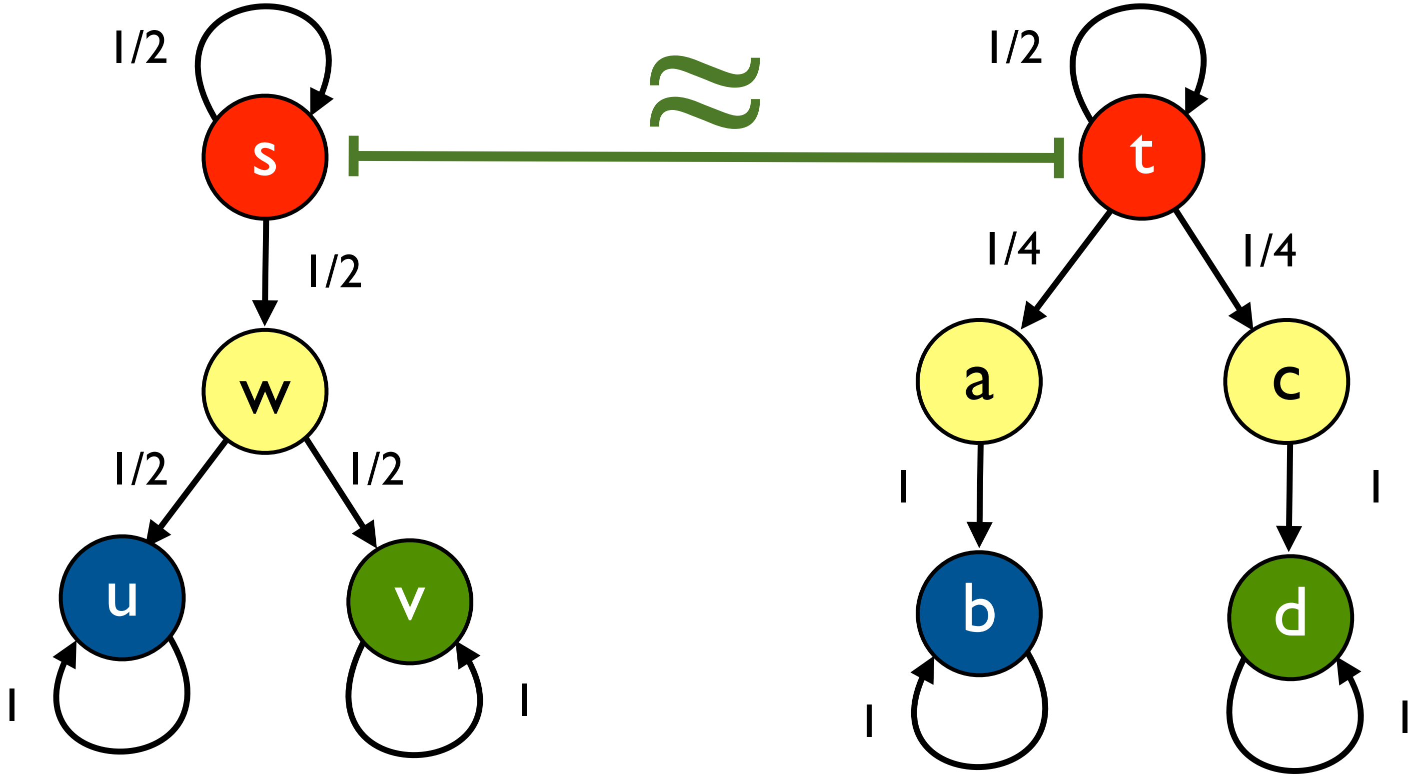
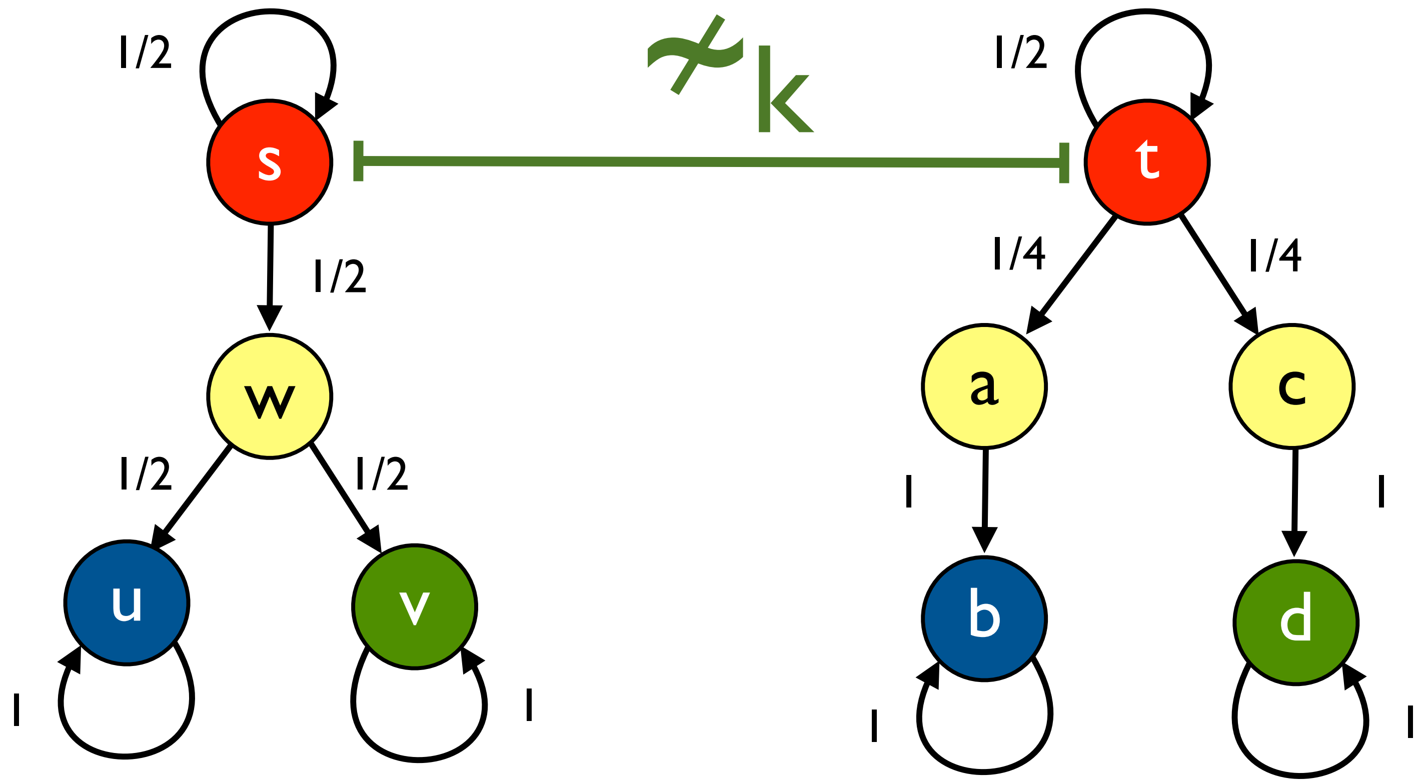
The Counterexample



The Counterexample



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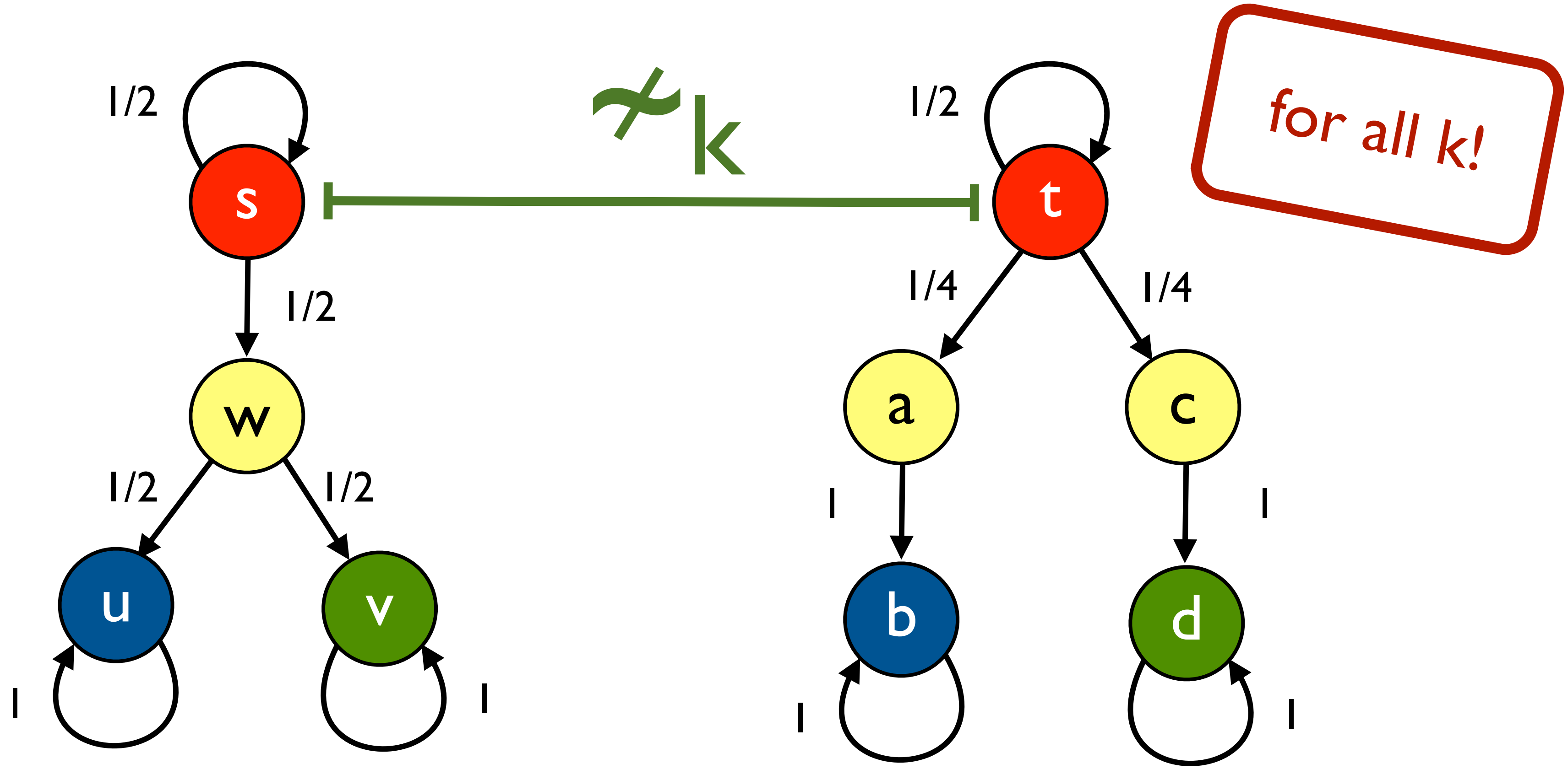

Concluding Remarks

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Future Work

- different kind of models (non-determinism?)
- logic distance parametric on sets of formulas
- explore topological properties

Thank you
for the attention