An Algebraic Theory of Markov Processes

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LICS'18

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Historical Perspective

- **Moggi'88:** *How to incorporate effects into denotational semantics?* -**Monads** as notions of computations

- **Plotkin & Power'01:** *(most of the) Monads are given by operations and equations* -**Algebraic Effects**

- **Hyland, Plotkin, Power'06:** *sum and tensor of theories* - **Combining Algebraic Effects**

- **Mardare, Panangaden, Plotkin (LICS'16):** *Theory of effects in a metric setting* -**Quantitative Algebraic Effects** (operations & *quantitative equations* give monads on **Met**)

$$s = t \quad \Longrightarrow \quad s =_{\varepsilon} t$$

Quantitative Equations

$$s =_{\varepsilon} t$$

"s is approximately equal to t up to an error ε"

What have we done

- Shown how to combine -by disjoint union- different theories to produce new interesting examples
- Specifically, equational axiomatization of Markov processes obtained by combining equations for transition systems and equations for probability distributions
- The equations are in the generalized quantitative sense of Mardare et al. LICS'16
- We have characterized the final coalgebra of Markov processes algebraically

Quantitative Equational Theory

Mardare, Panangaden, Plotkin (LICS'16)

A quantitative equational theory $\mathscr{U}$ of type Σ is a set of

$$\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s$$

conditional quantitative equations

closed under the following "meta axioms"

(Refl) $\vdash x =_0 x$

(Symm) $x =_\varepsilon y \vdash y =_\varepsilon x$

(Triang) $x =_\varepsilon y, y =_\delta z \vdash x =_{\varepsilon+\delta} y$

(NExp) $x_1 =_\varepsilon y_1, \ldots, y_n =_\varepsilon y_n \vdash f(x_1, \ldots, x_n) =_\varepsilon f(y_1, \ldots, y_n)$ – **for** $f \in \Sigma$

(Max) $x =_\varepsilon y \vdash x =_{\varepsilon+\delta} y$ – **for** $\delta > 0$

(Inf) $\{x =_\varepsilon y \mid \delta > \varepsilon\} \vdash x =_\varepsilon y$

(1-Bdd*) $\vdash x =_1 y$

Quantitative Algebras

Mardare, Panangaden, Plotkin (LICS'16)

The models of a quantitative equational theory $\mathcal{U}$ of type Σ are

Quantitative Σ-Algebras:

category of (1-bounded) metric spaces with non-expansive maps

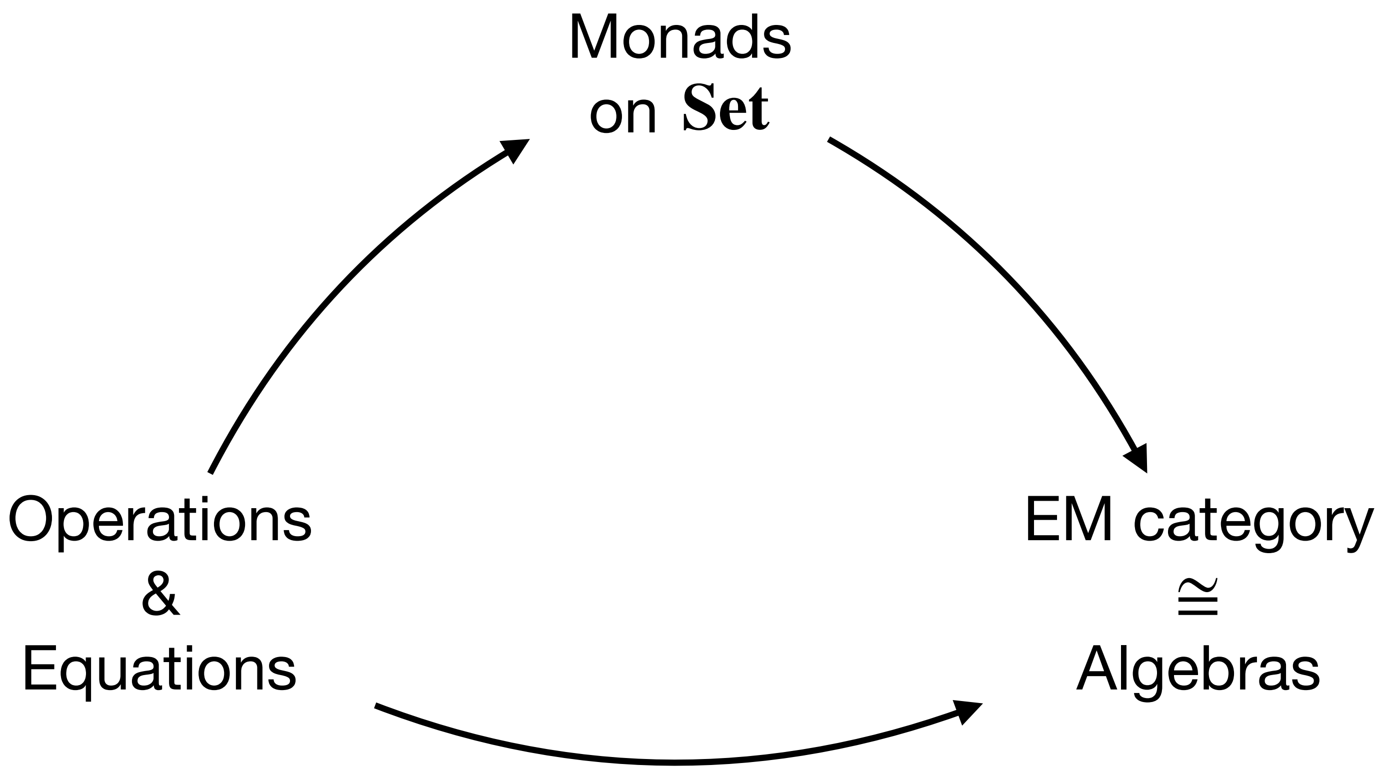
$\mathscr{A} = (A, \alpha\colon \Sigma A \to A)$ – **Universal Σ-algebras on Met**

Satisfying the all the quantitative equations in $\mathcal{U}$

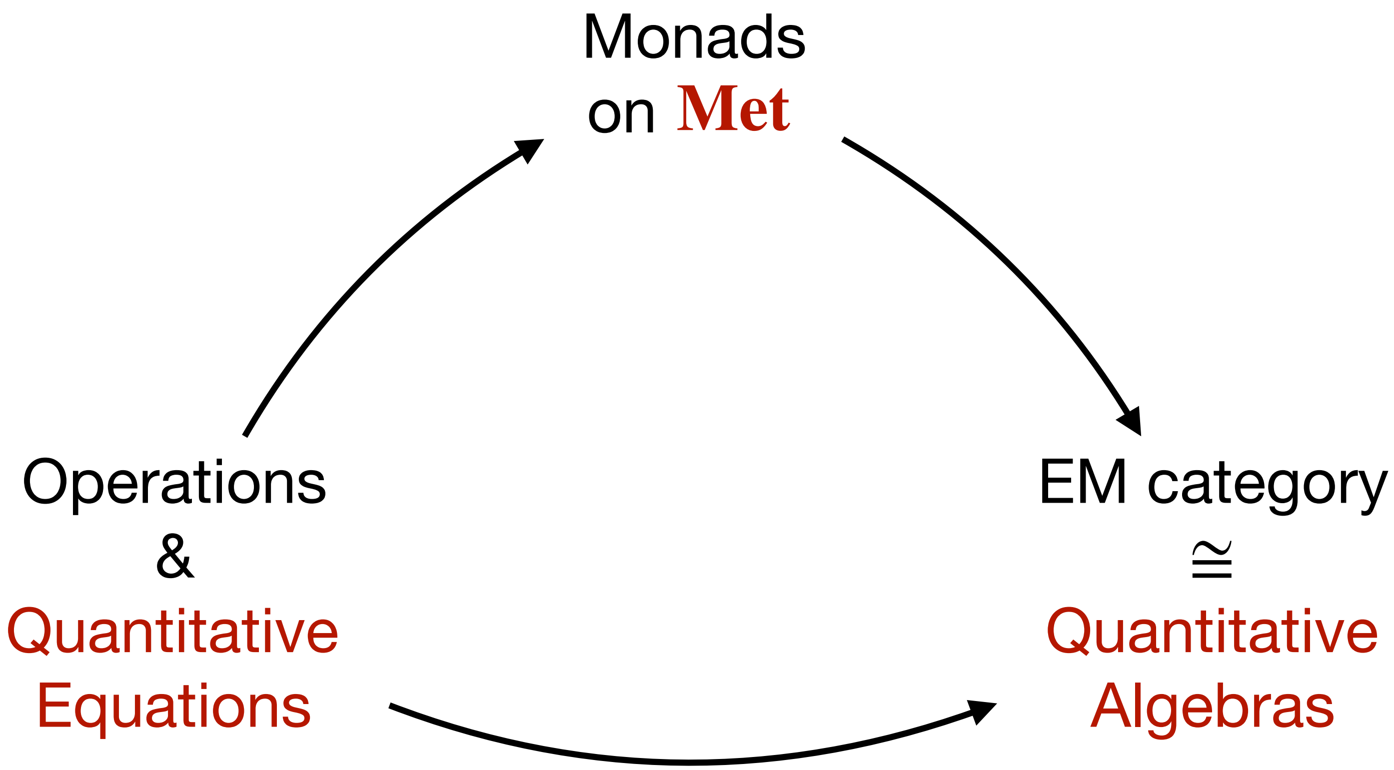
We denote the category of models of $\mathcal{U}$ by

$$\mathbb{K}(\Sigma, \mathcal{U})$$

Standard picture



Our picture


Monads
on **Met**
Operations
&
Quantitative
Equations
EM category
≅
Quantitative
Algebras

$\mathscr{U}$ Models are $T_{\mathscr{U}}$-Algebras

basic quantitative equation

$$\{x_i =_{\varepsilon_i} y_i \mid i \in I\} \vdash t =_{\varepsilon} s$$

A quantitative equational theory $\mathscr{U}$ is *basic* if it can be axiomatised by a set of basic conditional quantitative equations

Theorem

For any ***basic*** quantitative equational theory $\mathscr{U}$ of type Σ

$$\mathbb{K}(\Sigma, \mathscr{U}) \cong T_{\mathscr{U}}\textbf{-Alg}$$

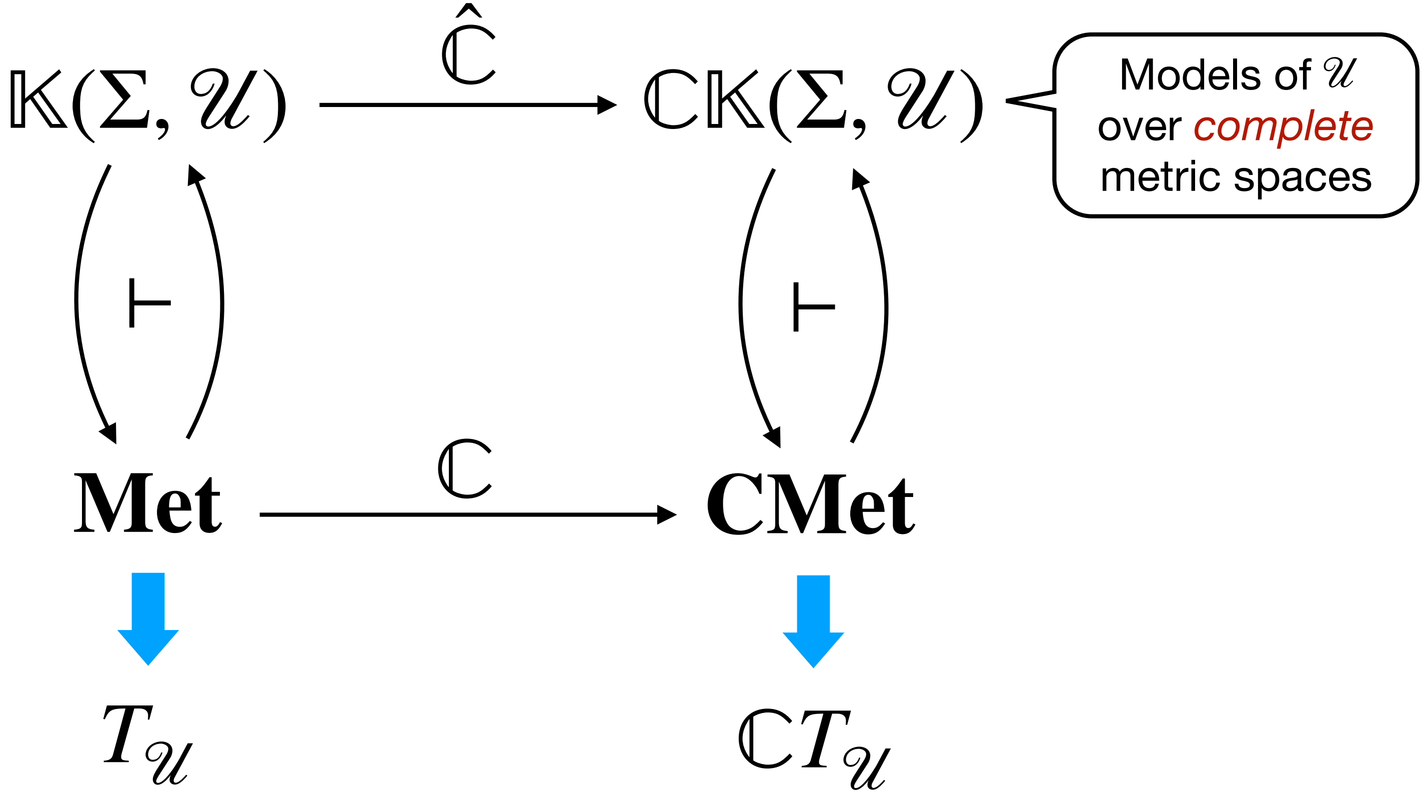
EM algebras for the monad $T_{\mathscr{U}}$

Free Monads on $\mathbf{CMet}$

A quantitative equational theory is *continuous* if it can be axiomatised by a collection of *continuous schemata* of quantitative equations

$$x_1 =_{\varepsilon_1} y_1, \ldots, x_n =_{\varepsilon_n} y_n \vdash t =_{\varepsilon} s \qquad \text{– for } \varepsilon \geq f(\varepsilon_1, \ldots, \varepsilon_n)$$

continuous real-valued function



Theory of Contractive Operators

arity

$$f\colon \langle n, {\color{red}c} \rangle \in \Sigma$$

(countable) Signature of contractive operators

contractive factor $0 < c < 1$

(*f*-Lip) $\{x_1 =_\varepsilon y_1, \ldots, y_n =_\varepsilon y_n\} \vdash f(x_1, \ldots, x_n) =_\delta f(y_1, \ldots, y_n)$ – **for** $\delta \geq c\varepsilon$

The theory $\mathcal{O}(\Sigma)$ induced by the axioms above is called *quantitative equational theory of contractive operators over* Σ

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Free monad on $\tilde{\Sigma} = \coprod_{f\colon \langle n,c\rangle \in \Sigma} (c \cdot Id)^n$

Monads

$$T_{\mathcal{O}(\Sigma)} \cong \tilde{\Sigma}^*$$

(on Met)

$$\mathbb{C}T_{\mathcal{O}(\Sigma)} \cong \tilde{\Sigma}^*$$

(on CSMet)

Interpolative Barycentric Theory

Mardare, Panangaden, Plotkin (LICS'16)

$$\Sigma_{\mathcal{B}} = \{ +_e : 2 \mid e \in [0,1] \}$$

(B1) $\vdash x +_1 y =_0 x$

(B2) $\vdash x +_e x =_0 x$

(SC) $\vdash x +_e y =_0 y +_{1-e} x$

(SA) $\vdash (x +_e y) +_d z =_0 x +_{ed} (y +_{\frac{d-ed}{1-ed}} z)$ – **for** $e, d \in [0,1)$

(IB) $x =_\varepsilon y, x' =_{\varepsilon'} y' \vdash x +_e x' =_\delta y +_e y'$ – **for** $\delta \geq e\varepsilon + (1-e)\varepsilon'$

The quantitative theory $\mathcal{B}$ induced by the axioms above is called *interpolative barycentric quantitative equational theory*

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Finitely supported Borel probability measures with **Kantorovich metric**

Monads

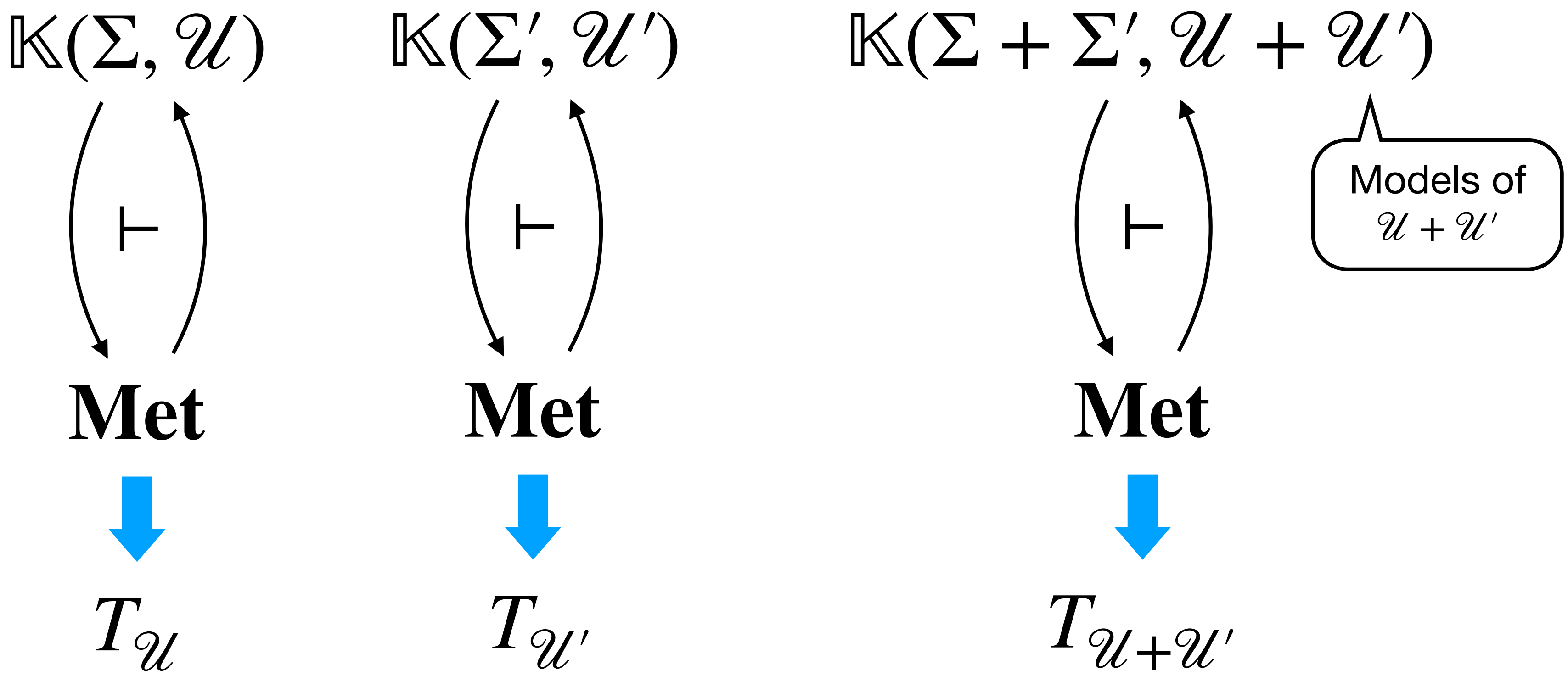
Borel probability measures with **Kantorovich metric** (Giry Monad)

$T_{\mathcal{B}} \cong \Pi$ **(on Met)**

$\mathbb{C}T_{\mathcal{B}} \cong \Delta$ **(on CSMet)**

Disjoint Union of Theories

The disjoint union $\mathscr{U} + \mathscr{U}'$ of two quantitative theories with disjoint signatures is the smallest quantitative theory containing $\mathscr{U}$ and $\mathscr{U}'$



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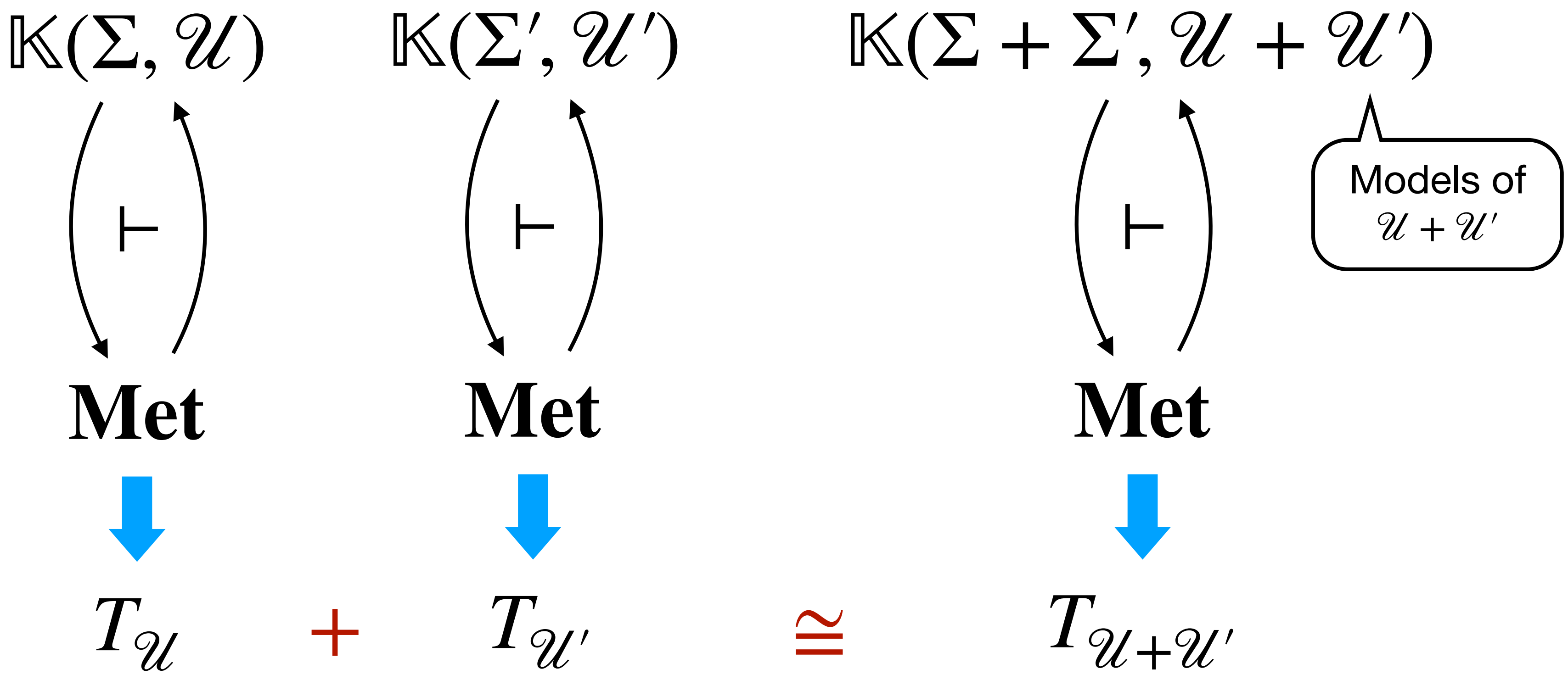

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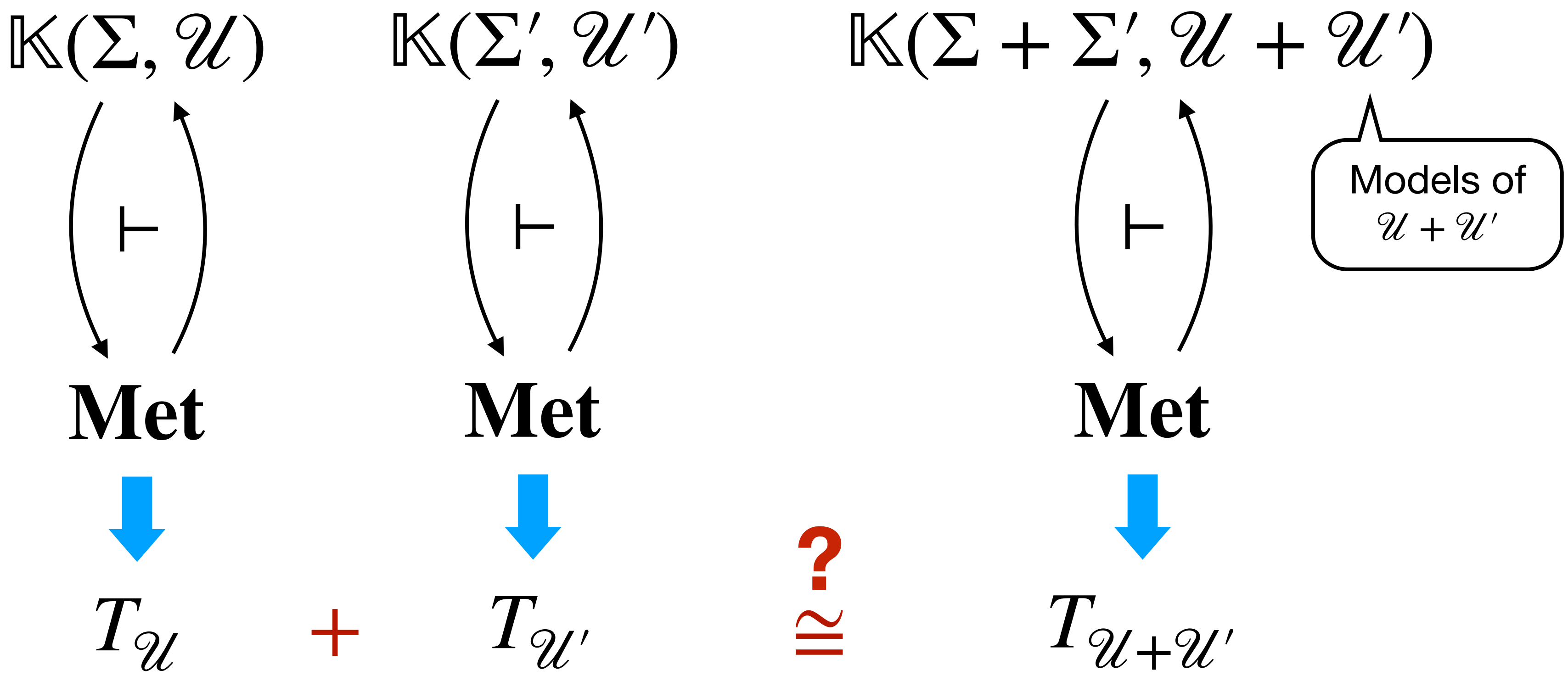

Disjoint Union of Theories

The answer is positive for ***basic*** quantitative theories

$$T_{\mathscr{U}} + T_{\mathscr{U}'} \cong T_{\mathscr{U}+\mathscr{U}'}$$

The proof follows standard techniques (Kelly'80)

Theorem

For ***basic*** quantitative equational theories $\mathscr{U}, \mathscr{U}'$ of type Σ, Σ'

$$\mathbb{K}(\Sigma + \Sigma', \mathscr{U} + \mathscr{U}') \cong \langle T_{\mathscr{U}}, T_{\mathscr{U}'} \rangle\textbf{-Alg} \cong (T_{\mathscr{U}} + T_{\mathscr{U}'})\textbf{-Alg}$$

EM-**bialgebras** for the monads $T_{\mathscr{U}}, T_{\mathscr{U}'}$

Interpolative Barycentric Theory with Contractive Operators

$$\Sigma_{\mathscr{B}} + \Sigma = \{ +_e : 2 \mid e \in [0,1] \} \cup \Sigma$$

$\mathscr{B}$

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$\mathscr{O}(\Sigma)$

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Monads

$T_{\mathscr{B}+\mathscr{O}(\Sigma)} \cong \Pi + \tilde{\Sigma}^*$ **(on Met)**

$\mathbb{C}T_{\mathscr{B}+\mathscr{O}(\Sigma)} \cong \Delta + \tilde{\Sigma}^*$ **(on CSMet)**

Sum with Free Monad

Hyland, Plotkin, Power (TCS 2016)

Theorem

For a functor F and a monad T, if the free monads F^* and $(FT)^*$ exist, then the sum of monads $T + F^*$ exists and is given by a canonical monad structure on the composite $T(FT)^*$

Corollary

Under same assumptions as above, the sum of monads $T + F^*$ is given by a canonical monad structure on $\mu y \,.\, T(Fy + -)$

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generalised resumption monad of (Cenciarelli, Moggi'93)

Markov Process Monads

We can recover a quantitative theory of Markov Processes as an interpolative barycentric theory with the following signature of operators

termination

transition to next state

$$\mathcal{M}_c = \{\mathbf{0} : \langle 0, c \rangle, \diamond : \langle 1, c \rangle\} \quad (\textbf{for } 0 < c < 1)$$

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Monads

Rooted acyclic finite Markov processes, with **c-probabilistic bisimilarity metric**

(on Met) $$T_{\mathscr{B}+\mathscr{O}(\mathscr{M}_c)} \cong \mu y . \Pi(1 + c \cdot y + -)$$

Markov processes on complete separable metric spaces with **c-probabilistic bisimilarity metric**

(on CSMet) $$\mathbb{C}T_{\mathscr{B}+\mathscr{O}(\mathscr{M}_c)} \cong \mu y . \Delta(1 + c \cdot y + -)$$

Final Coalgebra of MPs

$$\mathbb{C}T_{\mathscr{B}+\mathscr{O}(\mathscr{M}_c)} \cong \mu y\,.\,\Delta(1 + c \cdot y + -\ )$$

assigns to any $A \in \mathbf{CSMet}$ the initial solution of the equation

$$MP_A \cong \Delta(1 + c \cdot MP_A + A)$$

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Theorem (Turi, Rutten'98)

Every *locally contractive functor* H on $\mathbf{CMet}$ has a unique fixed point, which is both an *initial algebra* and a *final coalgebra for* H

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Every *locally contractive functor* H on $\mathbf{CMet}$ has a unique fixed point, which is both an *initial algebra* and a *final coalgebra for* H

In particular, when $A \in \mathbf{0}$ (the empty metric space)

$$MP_{\mathbf{0}} \to \Delta(1 + c \cdot MP_{\mathbf{0}})$$

final coalgebra of Markov processes

Conclusions

- Sum of quantitative theories (this opens the way to developing combinations of quantitative effects)

- Unifying algebraic and coalgebraic presentation of Markov processes (coincidence with initial and final coalgebra)

- Tensor product of quantitative theories?