

Complete Axiomatization for the Total Variation Distance of MCs

Giorgio Bacci, Giovanni Bacci,
Kim G. Larsen, and Radu Mardare

Aalborg University, DK

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Introduction

- **Kleene's Theorem:** fundamental correspondence between regular expressions and DFAs
- **Salomaa'66, Kozen'91:** complete axiomatization for proving equivalence of regular expressions
- **Milner'83:** applied the above program on process behaviors and LTSs
- **Rabinovich'83:** distributivity axiom capturing trace equivalence of LTSs
- Many variations of the above schema

Example: Markov chains

Expressions: $t, s ::= X \mid a.t \mid t +_e s \mid \text{rec } X.t$

Example: Markov chains

names
 $X \in \mathbb{X}$

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The diagram illustrates the components of the expression grammar. A callout box labeled "names $X \in \mathbb{X}$ " points to the variable X . Another callout box labeled "action-prefix $a \in \mathbb{A}$ " points to the action prefix $a.t$.

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probabilistic choice
 $e \in [0,1]$

action-prefix
 $a \in \mathbb{A}$

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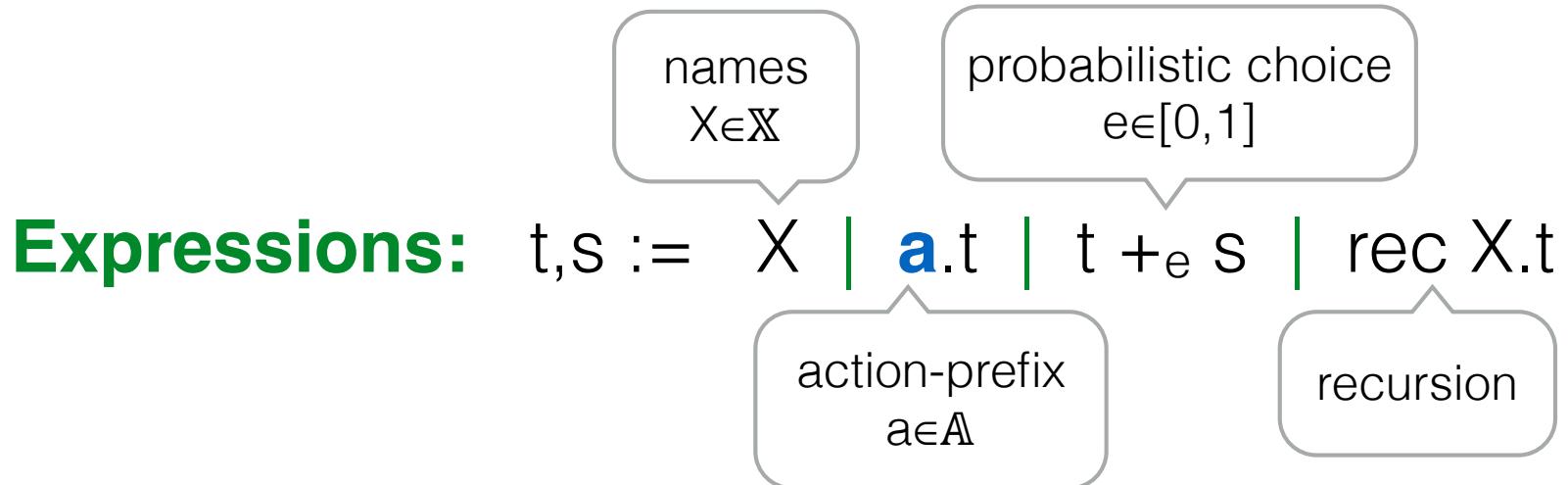
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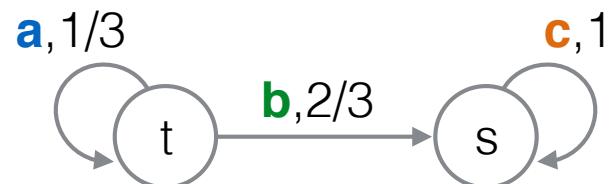
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recursion

Example: Markov chains



Kleene's theorem for MCs



$$t = \text{rec } X.(\mathbf{a}.X +_{1/3} \mathbf{b}.s)$$
$$s = \text{rec } Y.(\mathbf{c}.Y)$$

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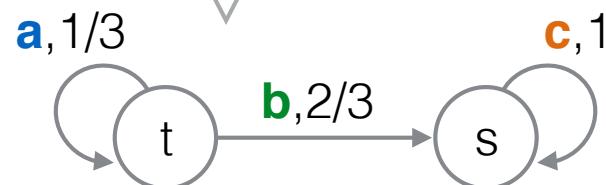
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finite MCs



Kleene's theorem for MCs

$$\begin{aligned} t &= \text{rec } X.(\mathbf{a}.X +_{1/3} \mathbf{b}.s) \\ s &= \text{rec } Y.(\mathbf{c}.Y) \end{aligned}$$

Example: Markov chains

(B1) $\vdash t +_1 s = t$

(B2) $\vdash t +_e t = t$

(SC) $\vdash t +_e s = s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u = t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e, e' \in [0, 1]$

(Unfold) $\vdash \text{rec } X.t = t[\text{rec } X.t / X]$

(Fix) $\{t = s[t / X]\} \vdash t = \text{rec } X.s$ — for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) = \text{rec } X.t$

(Dist-pref) $\vdash a.(t +_e s) = a.t +_e a.s$

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Silva-Sokolova axiomatization
probabilistic trace equivalence (MFPS'11)

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Rabinovich's distributivity axiom

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...for probabilistic systems

- **Generative Markov chains:**

Baeten-Bergstra-Smolka'95 & Stark-Smolka'00

- **Simple Probabilistic Automata:**

Bandini-Segala'01

- **(fully) Probabilistic Automata:**

Mislove-Ouaknine-Worrell'04 (strong-bisimulation)

Deng-Palamidessi'07 (weak-bisimulation & behavioral eq.)

- **Quantitative Kleene Coalgebras:**

Silva-Bonchi-Bonsangue-Rutten'11 (coagebraic bisim.)

- etc...

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- How do we do it?
By using **Quantitative Equational Theories*** of
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- How do we do it?
By using **Quantitative Equational Theories*** of
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$$s = t \quad \longrightarrow \quad s =_{\varepsilon} t$$

Equational Theories

$$\{t_i = s_i \mid i \in I\} \vdash t = s$$

inference

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inference

(Refl) $\vdash t = t$

(Symm) $\{t = s\} \vdash s = t$

(Trans) $\{t = u, u = s\} \vdash t = s$

(Cong) $\{t_1 = s_1, \dots, t_n = s_n\} \vdash f(t_1, \dots, t_n) = f(s_1, \dots, s_n)$ — for $f \in \Sigma$

Quantitative Theories

Mardare-Panangaden-Plotkin (LICS'16)

$$\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s$$

quantitative
inference

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_{\varepsilon} s\} \vdash s =_{\varepsilon} t$

(Triang) $\{t =_{\varepsilon} u, u =_{\delta} s\} \vdash t =_{\varepsilon+\delta} s$

(NExp) $\{t_1 =_{\varepsilon} s_1, \dots, t_n =_{\varepsilon} s_n\} \vdash f(t_1, \dots, t_n) =_{\varepsilon} f(s_1, \dots, s_n) \quad \text{for } f \in \Sigma$

(Max) $\{t =_{\varepsilon} s\} \vdash t =_{\varepsilon+\delta} s \quad \text{for } \delta > 0$

(Arch) $\{t =_{\delta} s \mid \delta > \varepsilon\} \vdash t =_{\varepsilon} s$

Quantitative Semantics

Quantitative Algebra

$$\mathcal{A} = (A, \Sigma_A, d_A) \begin{array}{l} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \begin{array}{l} (A, \Sigma_A) \text{ — Universal algebra} \\ (A, d_A) \text{ — (pseudo)metric space} \end{array}$$

Satisfiability

$$\mathcal{A} \models \left(\{t_i =_{\varepsilon_i} s_i \mid i \in I\} \vdash t =_{\varepsilon} s \right)$$

iff

for all $i \in I$. $d_A([t_i], [s_i]) \leq \varepsilon_i$ implies $d_A([t], [s]) \leq \varepsilon$

completeness

quantitative
algebra

quantitative
theory

$$\mathcal{A} \models (\vdash t =_{\varepsilon} s) \quad (\vdash t =_{\varepsilon} s) \in \mathcal{U}$$

soundness

quantitative
algebra

completeness

quantitative
theory

$$\mathcal{A}_{\text{MC}} \models (\vdash t =_{\varepsilon} s) \quad (\vdash t =_{\varepsilon} s) \in \mathcal{U}_{\text{MC}}$$

soundness

The Quantitative Universal Algebra

Universal Algebra of MCs

Signature: $X : 0 \mid a_{\cdot -} : 1 \mid +_e : 2 \mid \text{rec } X : 1$

$$(X)_{\text{MC}} = \boxed{X}$$

$$(a_{\cdot -} m)_{\text{MC}} = \begin{array}{c} a.m \\ \downarrow \\ m \end{array}$$

$$\begin{aligned} & (m +_e n)_{\text{MC}} = \begin{array}{c} m+n \\ e\mu + (1-e)\nu \\ \downarrow \\ m + \mathcal{N} \end{array} \\ & (\text{rec } X. m)_{\text{MC}} = \begin{array}{c} \text{rec } X. m \\ \mu \\ \downarrow \\ m \end{array} \end{aligned}$$

Total variation distance

$$tv(m,n) = \sup_{E \in \sigma(\Pi)} | P(m)(E) - P(n)(E) |$$

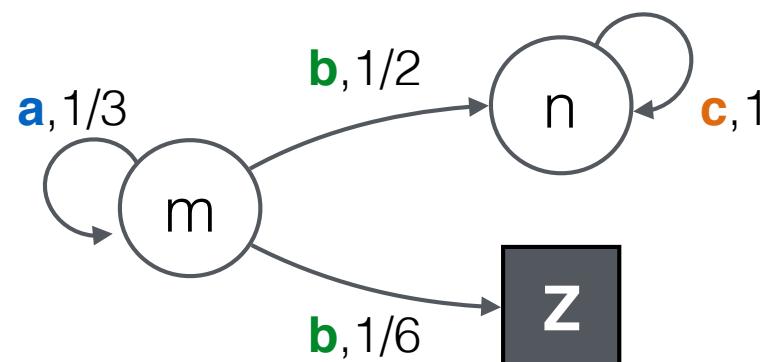
$$\Pi = A^\omega \cup A^* X \quad (\textit{observable traces})$$

ω -traces over A

finite traces over A terminating in X

$$P(m)(\mathfrak{C}(aab\mathbf{c})) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1$$

$$P(m)(\mathfrak{C}(aabZ)) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{6}$$

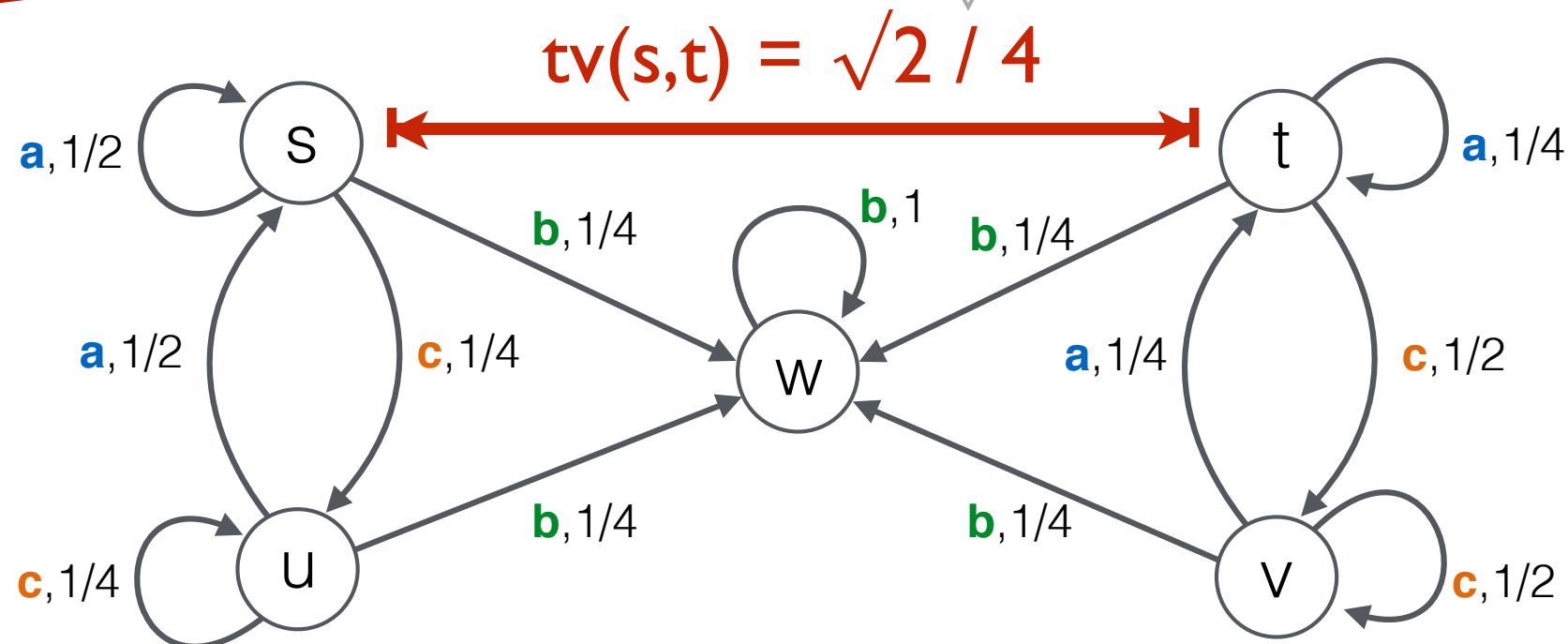


A tiny yet tricky example

(from Chen-Kiefer LICS'14)

maximizing event
is not ω -regular!

irrational number



Converging to Total Variation

(Bacci et al. ICTAC'15)

\mathbb{K} — poset of positive integers ordered by divisibility

Theorem

1. The net $(d_k)_{k \in \mathbb{K}}$ converges point-wise to \mathbf{tv}
2. for all $k \in \mathbb{K}$, $d_k \geq \mathbf{tv}$

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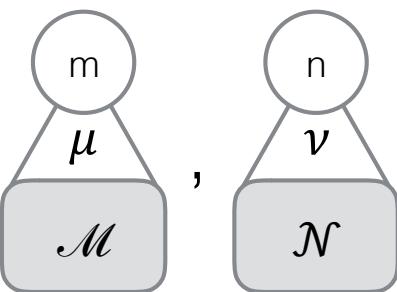
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probabilistic k-bisimilarity distance

k -bisimilarity distance

(generalizes Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

$$d_k(\mu, \nu) = \min \{ \int \Lambda(d_k) d\omega \mid \omega \in \Omega(\mu^k, \nu^k) \}$$


The diagram illustrates the components of the k-bisimilarity distance. It shows two nodes, m and n, each connected to a distribution. Node m is connected to distribution μ , which is represented by a gray trapezoid labeled \mathcal{M} . Node n is connected to distribution ν , which is represented by a gray trapezoid labeled \mathcal{N} .

k -bisimilarity distance

(generalizes Desharnais et al. TCS'04)

it is the least 1-bounded pseudometric satisfying

$$d_k(\begin{array}{c} m \\ \mu \\ \mathcal{M} \end{array}, \begin{array}{c} n \\ \nu \\ \mathcal{N} \end{array}) = \min \left\{ \int \Lambda(d_k) d\omega \mid \omega \in \Omega(\mu^k, \nu^k) \right\}$$


$\Lambda(d_k)$ —greatest 1-bounded pseudometric on $(\mathbb{A}^k \times MC) \cup \mathbb{A}^{<k} \times$

$$\text{s.t., for all } w \in \mathbb{A}^k, \quad \Lambda(d_k)((w, \begin{array}{c} m \\ \mu \\ \mathcal{M} \end{array}), (w, \begin{array}{c} n \\ \nu \\ \mathcal{N} \end{array})) = d_k(\begin{array}{c} m \\ \mu \\ \mathcal{M} \end{array}, \begin{array}{c} n \\ \nu \\ \mathcal{N} \end{array})$$

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s.t, for all $w \in \mathbb{A}^k$, $\Lambda(d_k)((w, \begin{array}{c} m \\ \mu \\ \mathcal{M} \end{array}), (w, \begin{array}{c} n \\ \nu \\ \mathcal{N} \end{array})) = d_k(\begin{array}{c} m \\ \mu \\ \mathcal{M} \end{array}, \begin{array}{c} n \\ \nu \\ \mathcal{N} \end{array})$

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Kantorovich lifting

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couplings
= probabilistic “relations”

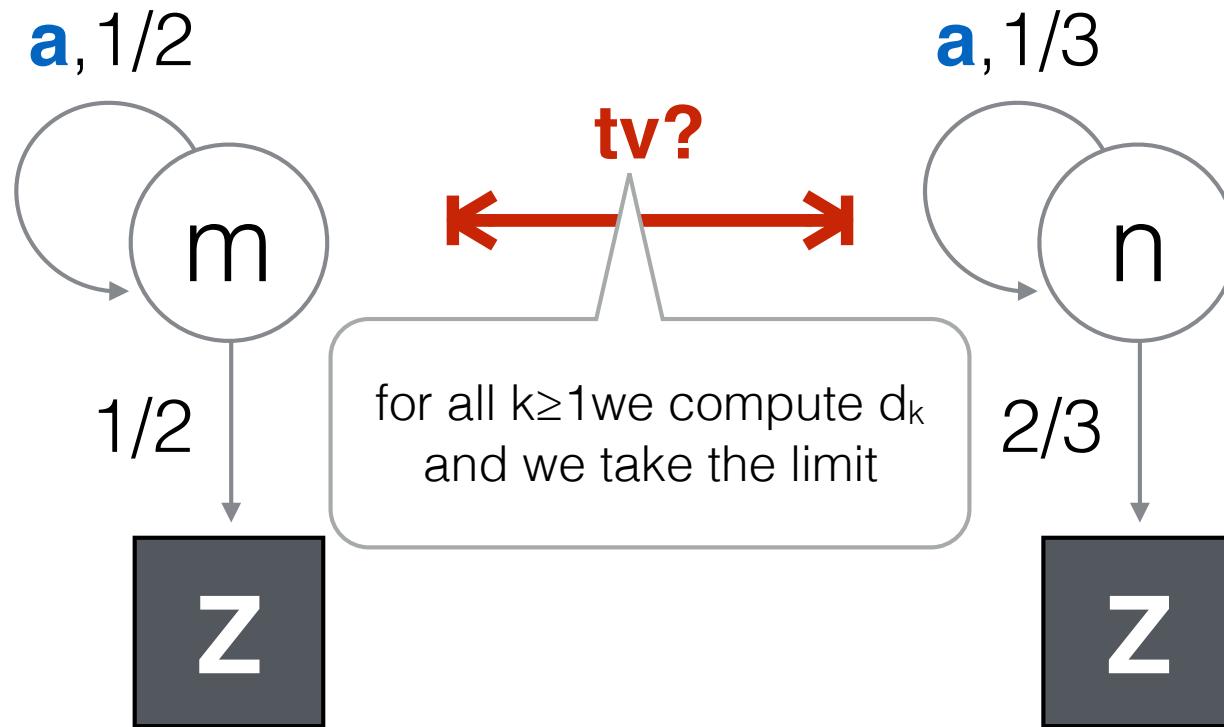
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$$\text{s.t., for all } w \in \mathbb{A}^k, \quad \Lambda(d_k)((w, m), (w, n)) = d_k(m, n)$$

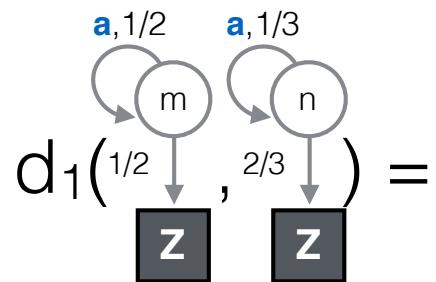
Running example



$$m = \text{rec } X. (\mathbf{a}.X +_{1/2} Z)$$

$$n = \text{rec } Y. (\mathbf{a}.Y +_{1/3} Z)$$

optimal coupling between
transition probabilities
of m and n



ω^*

$\nu((\mathbf{a},n))$	$\nu(Z)$
$1/3$	$2/3$
$1/3$	$1/6$

$\mu((\mathbf{a},m)) = 1/2$

$\mu(Z) = 1/2$

optimal coupling between
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$$d_1\left(\begin{matrix} \text{a}, 1/2 \\ m \\ z \end{matrix}, \begin{matrix} \text{a}, 1/3 \\ n \\ z \end{matrix}\right) =$$

$$= \frac{1}{3} \Lambda(d_1)((\text{a}, \begin{matrix} 1/2 \\ m \\ z \end{matrix}), (\text{a}, \begin{matrix} 2/3 \\ n \\ z \end{matrix})) + \frac{1}{6} \Lambda(d_1)((\text{a}, \begin{matrix} 1/2 \\ m \\ z \end{matrix}), [\text{z}]) + \frac{1}{2} \Lambda(d_1)([\text{z}], [\text{z}])$$

optimal coupling between transition probabilities of m and n

ω^*

$\nu((\text{a}, n))$	$\nu(Z)$
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1/3	1/6
	1/2

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$$\mu(Z) = \mathbf{1/2}$$

$$\omega^* \quad \begin{matrix} \nu((\text{a}, n)) \\ \mathbf{1/3} \end{matrix} \quad \begin{matrix} \nu(Z) \\ \mathbf{2/3} \end{matrix}$$

	$\nu((\text{a}, n))$	$\nu(Z)$
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		$1/2$

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$$d_1\left(\begin{matrix} \text{a}, 1/2 \\ m \end{matrix}, \begin{matrix} \text{a}, 1/3 \\ n \end{matrix}\right) =$$

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$$\mu((\mathbf{a}, m)) = \mathbf{1/2}$$

$$\mu(Z) = \mathbf{1/2}$$

ω^*

$\nu((\mathbf{a}, n))$	$\nu(Z)$
$\mathbf{1/3}$	$\mathbf{2/3}$

1/3	1/6
	1/2

optimal coupling between
transition probabilities
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$$d_1\left(\begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}, \begin{matrix} \text{a}, 1/3 \\ 2/3 \end{matrix}\right) =$$

$$\begin{aligned}
 &= \frac{1}{3} \wedge(d_1)\left(\left(\text{a}, \frac{1}{2}\right),\left(\text{a}, \frac{2}{3}\right)\right) + \frac{1}{6} \wedge(d_1)\left(\left(\text{a}, \frac{1}{2}\right), z\right) + \frac{1}{2} \wedge(d_1)(z, z) \\
 &= \frac{1}{3} d_1\left(\begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}, \begin{matrix} \text{a}, 1/3 \\ 2/3 \end{matrix}\right) + \frac{1}{6}
 \end{aligned}$$

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$$\mu(Z) = \mathbf{1/2}$$

$$\omega^* \quad \begin{array}{c} \nu((\mathbf{a}, n)) \\ \mathbf{1/3} \end{array} \quad \begin{array}{c} \nu(Z) \\ \mathbf{2/3} \end{array}$$

	$\nu((\mathbf{a}, n))$	$\nu(Z)$
$\mathbf{1/3}$	$1/3$	$1/6$
		$1/2$

optimal coupling between
transition probabilities
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$$= \frac{1}{3} \wedge(d_1)\left(\begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}, \begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}\right), \left(\begin{matrix} \text{a}, 1/3 \\ 2/3 \end{matrix}, \begin{matrix} \text{a}, 1/3 \\ 2/3 \end{matrix}\right) \right) + \frac{1}{6} \wedge(d_1)\left(\begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}, \begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}\right), \left(\begin{matrix} \text{z} \\ \text{z} \end{matrix}, \begin{matrix} \text{z} \\ \text{z} \end{matrix}\right) \right) + \frac{1}{2} \wedge(d_1)\left(\begin{matrix} \text{z} \\ \text{z} \end{matrix}, \begin{matrix} \text{z} \\ \text{z} \end{matrix}\right), \left(\begin{matrix} \text{z} \\ \text{z} \end{matrix}, \begin{matrix} \text{z} \\ \text{z} \end{matrix}\right) \right)$$

$$= \frac{1}{3} d_1\left(\begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}, \begin{matrix} \text{a}, 1/3 \\ 2/3 \end{matrix}\right) + \frac{1}{6}$$

Solution: $d_1\left(\begin{matrix} \text{a}, 1/2 \\ 1/2 \end{matrix}, \begin{matrix} \text{a}, 1/3 \\ 2/3 \end{matrix}\right) = \frac{1}{4}$

ω^* $\nu((\text{a}, n))$ $\nu(Z)$
 1/3 $2/3$

$\mu((\text{a}, m)) = 1/2$

$\mu(Z) = 1/2$

1/3	1/6
	1/2

general case ($k \geq 1$)

$$d_k\left(\begin{array}{c} a, 1/2 \\ 1/2 \end{array}, \begin{array}{c} a, 1/3 \\ 2/3 \end{array}\right) = \frac{1}{3^k} d_k\left(\begin{array}{c} a, 1/2 \\ 1/2 \end{array}, \begin{array}{c} a, 1/3 \\ 2/3 \end{array}\right) + \frac{1}{6}$$

Solution:

$$d_k\left(\begin{array}{c} a, 1/2 \\ 1/2 \end{array}, \begin{array}{c} a, 1/3 \\ 2/3 \end{array}\right) = \frac{3^{k-1}}{2(3^k - 1)}$$

by the convergence theorem...

$$tv\left(\begin{array}{c} a, 1/2 \\ 1/2 \end{array}, \begin{array}{c} a, 1/3 \\ 2/3 \end{array}\right) = \lim_{k \rightarrow \infty} \frac{3^{k-1}}{2(3^k - 1)} = \frac{1}{6}$$

The Quantitative Equational Theory

rec is problematic...

The quantitative equational framework
of Mardare-Panangaden-Plotkin requires
all operators to be **non-expansive**

(NExp) $\{t_1 =_\varepsilon s_1, \dots, t_n =_\varepsilon s_n\} \vdash f(t_1, \dots, t_n) =_\varepsilon f(s_1, \dots, s_n)$ — for $f \in \Sigma$

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... but the NExp axiom is not sound for recursion

$\mathcal{A}_{MC} \not\models (\{t =_\varepsilon s\} \vdash \text{rec } X.t =_\varepsilon \text{rec } X.s)$

(see paper for the counterexample)

Relaxing non-expansivity

we keep all the axioms of quantitative algebras
but the NExp axiom

(Refl) $\vdash t =_0 t$

(Symm) $\{t =_\varepsilon s\} \vdash s =_\varepsilon t$

(Triang) $\{t =_\varepsilon u, u =_\delta s\} \vdash t =_{\varepsilon+\delta} s$

(NExp) $\{t_1 =_\varepsilon s_1, \dots, t_n =_\varepsilon s_n\} \vdash f(t_1, \dots, t_n) =_\varepsilon f(s_1, \dots, s_n)$ — for $f \in \Sigma$

(Max) $\{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta > 0$

(Arch) $\{t =_\delta s \mid \delta > \varepsilon\} \vdash t =_\varepsilon s$

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is NOT the original
quantitative equational
framework!

$$(\text{Max}) \{t =_\varepsilon s\} \vdash t =_{\varepsilon+\delta} s \quad \text{for } \delta > 0$$

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quantitative equational
framework!

the Archimedean axiom will be used
to recover completeness

Axiomatization

(B1) $\vdash t +_1 s =_0 t$

Interpolative barycentric axioms
(Mardare-Panangaden-Plotkin LICS'16)

(B2) $\vdash t +_e t =_0 t$

(SC) $\vdash t +_e s =_0 s +_{1-e} t$

(SA) $\vdash (t +_e s) +_{e'} u =_0 t +_{ee'} (s +_{\frac{e'-ee'}{1-ee'}} u)$ — for $e, e' \in [0, 1]$

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$ — for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

(Top) $\vdash t =_1 s$

(Unfold) $\vdash \text{rec } X.t =_0 t[\text{rec } X.t / X]$

Milner's recursion axioms

(Fix) $\{t = s[t / X]\} \vdash t =_0 \text{rec } X.s$

— for X guarded in t

(Unguard) $\vdash \text{rec } X.(t +_e X) =_0 \text{rec } X.t$

(Dist-pref) $\vdash \mathbf{a}.(t +_e s) = \mathbf{a}.t +_e \mathbf{a}.s$

Rabinovich's distributivity axiom

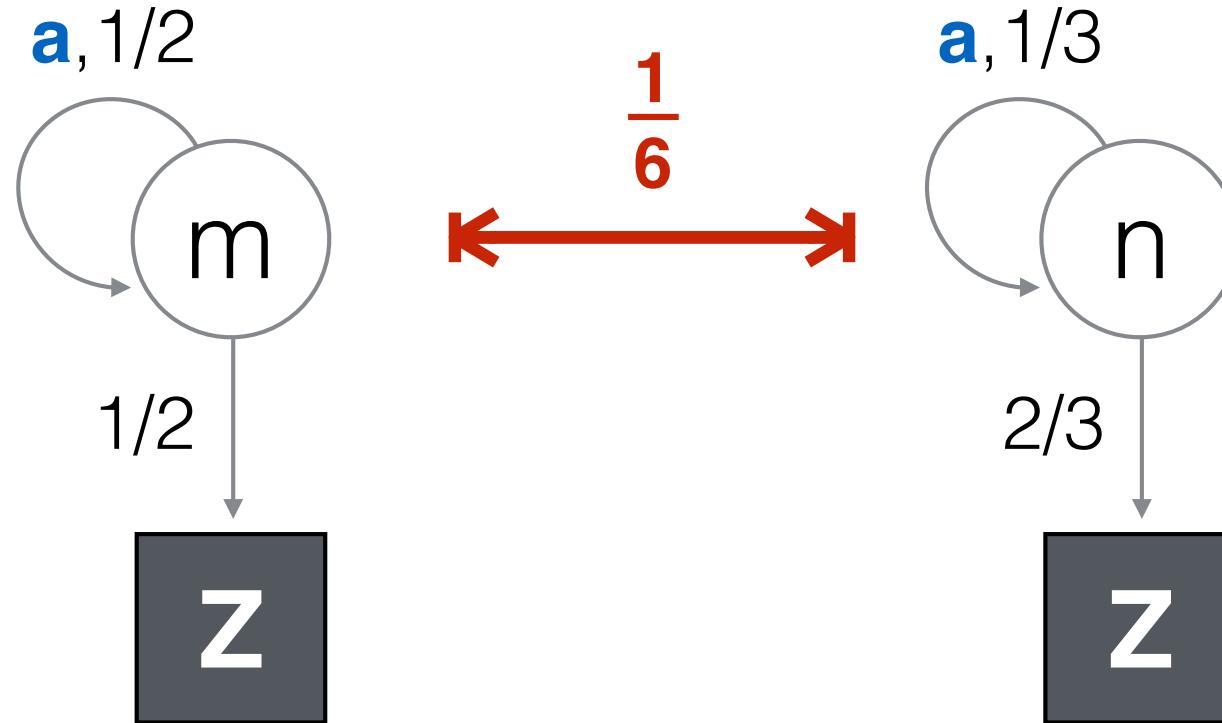
(Pref) $\{t =_\varepsilon s\} \vdash \mathbf{a}.t =_\varepsilon \mathbf{a}.s$

Weak NExp-axioms

(Cong) $\{t =_0 s\} \vdash \text{rec } X.t =_0 \text{rec } X.s$

proof of completeness (by example)

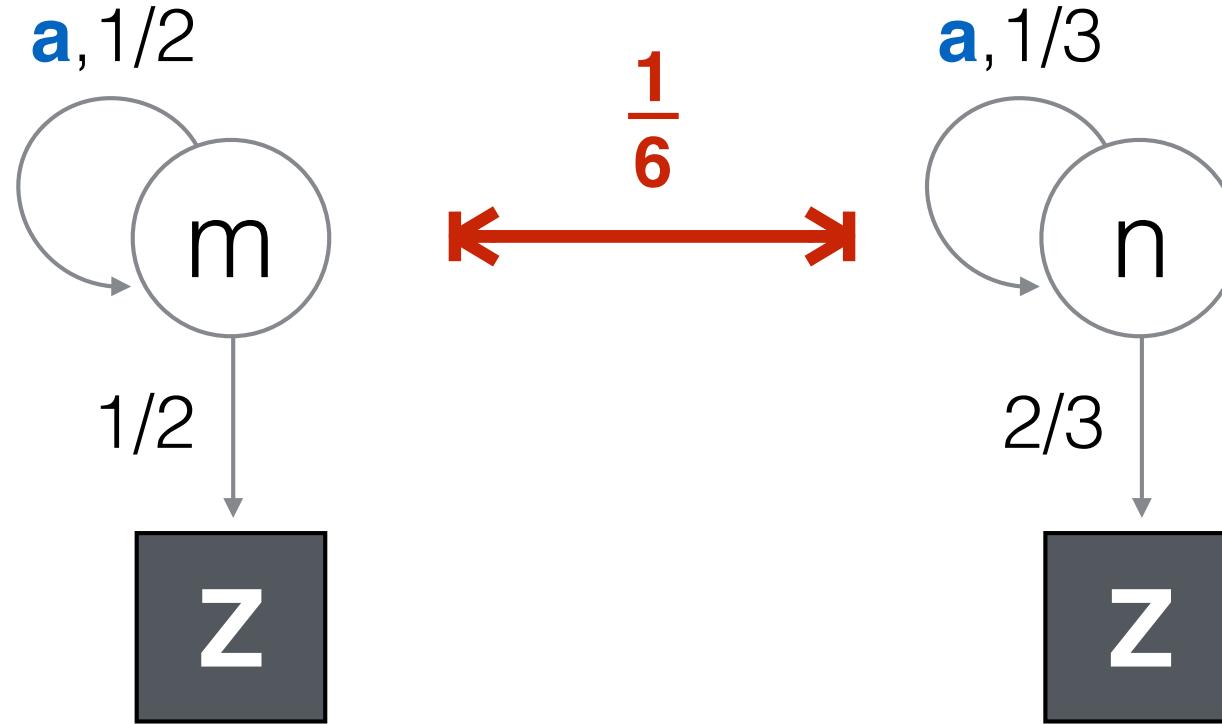
Same running example



$$m = \text{rec } X. (a.X +_{1/2} Z)$$

$$n = \text{rec } Y. (a.Y +_{1/3} Z)$$

Same running example



$$m = \text{rec } X. (a.X +_{1/2} Z)$$

$$n = \text{rec } Y. (a.Y +_{1/3} Z)$$

by using (Fix)+(Unfold)...

$$m = a.m +_{1/2} Z$$

$$n = a.n +_{1/3} Z$$

Kantorovich via IB

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$
— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

Kantorovich via IB

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$
— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

$$\begin{aligned} \mathbf{a.m} +_{1/2} Z &=_0 (\mathbf{a.m} +_{1/3} \mathbf{a.m}) +_{1/2} Z \quad (\text{B2}) \\ &=_0 \mathbf{a.m} +_{1/6} (\mathbf{a.m} +_{2/5} Z) \quad (\text{SA}) \end{aligned}$$

Kantorovich via IB

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$
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$$\begin{aligned} \mathbf{a}.m +_{1/2} Z &=_0 (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z & (\text{B2}) \\ &=_0 \mathbf{a}.m +_{1/6} (\mathbf{a}.m +_{2/5} Z) & (\text{SA}) \end{aligned}$$

$$\begin{aligned} \mathbf{a}.n +_{1/3} Z &=_0 Z +_{2/3} \mathbf{a}.n & (\text{SC}) \\ &=_0 (Z +_{1/4} Z) +_{2/3} \mathbf{a}.n & (\text{B2}) \\ &=_0 Z +_{1/6} (\mathbf{a}.n +_{2/5} Z) & (\text{SA}) + (\text{SC}) \end{aligned}$$

Kantorovich via IB

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$
— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

$$\mathbf{a}.m +_{1/2} Z =_0 (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z \quad (\text{B2})$$

$$=_0 \mathbf{a}.m +_{1/6} (\mathbf{a}.m +_{2/5} Z) \quad (\text{SA})$$

ω^*	(\mathbf{a}, Y)	Z
(\mathbf{a}, X)	1/3	1/6
Z		1/2

$$\mathbf{a}.n +_{1/3} Z =_0 Z +_{2/3} \mathbf{a}.n \quad (\text{SC})$$

$$=_0 (Z +_{1/4} Z) +_{2/3} \mathbf{a}.n \quad (\text{B2})$$

$$=_0 Z +_{1/6} (\mathbf{a}.n +_{2/5} Z) \quad (\text{SA}) + (\text{SC})$$

Kantorovich via IB

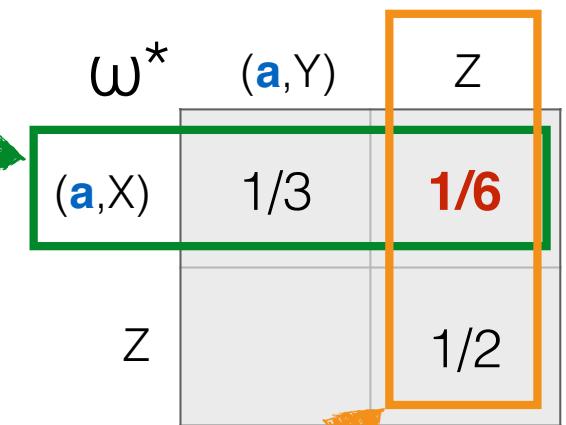
(IB) $\{t =_{\varepsilon} s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_{\delta} s +_e s'$
 — for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

$$\begin{aligned}
 \mathbf{a}.m +_{1/2} Z &=_0 (\mathbf{a}.m +_{1/3} \mathbf{a}.m) +_{1/2} Z && \text{(B2)} \\
 &=_0 \mathbf{a}.m + \mathbf{1/6} (\mathbf{a}.m +_{2/5} Z) && \text{(SA)} \rightarrow \begin{array}{c} \omega^* \\ \hline \begin{matrix} & (\mathbf{a},Y) & Z \\ \hline (\mathbf{a},X) & 1/3 & \mathbf{1/6} \\ \hline & & \end{matrix} \end{array} \\
 \mathbf{a}.n +_{1/3} Z &=_0 Z +_{2/3} \mathbf{a}.n && \text{(SC)} \\
 &=_0 (Z +_{1/4} Z) +_{2/3} \mathbf{a}.n && \text{(B2)} \\
 &=_0 Z + \mathbf{1/6} (\mathbf{a}.n +_{2/5} Z) && \text{(SA)+(SC)}
 \end{aligned}$$

Kantorovich via IB

(IB) $\{t =_\varepsilon s, t' =_{\varepsilon'} s'\} \vdash t +_e t' =_\delta s +_e s'$
— for $\delta \leq e\varepsilon + (1-e)\varepsilon'$

$$\begin{aligned} a.m +_{1/2} Z &=_0 (a.m +_{1/3} a.m) +_{1/2} Z && (B2) \\ &=_0 a.m + \textcolor{red}{1/6} (a.m +_{2/5} Z) && (SA) \end{aligned}$$



$$\begin{aligned} a.n +_{1/3} Z &=_0 Z +_{2/3} a.n && (SC) \\ &=_0 (Z +_{1/4} Z) +_{2/3} a.n && (B2) \\ &=_0 Z + \textcolor{red}{1/6} (a.n +_{2/5} Z) && (SA)+(SC) \end{aligned}$$

from what we have seen before and
(Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{\mathbf{1/3\varepsilon+1/6}} n$$

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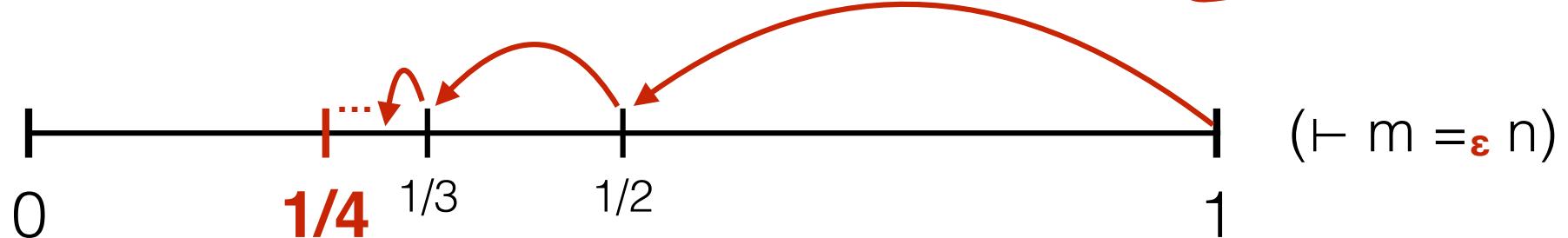
greatest
fixed point
operator

from what we have seen before and
 (Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{1/3\varepsilon+1/6} n$$

$$(\textbf{Top}) \vdash m =_1 n$$

greatest
fixed point
operator



$$(\textbf{Max}) \{t =_{\varepsilon} s\} \vdash t =_{\varepsilon+\delta} s \quad \text{--- for } \delta > 0$$

$$(\textbf{Arch}) \{t =_{\delta} s \mid \delta > \varepsilon\} \vdash t =_{\varepsilon} s$$

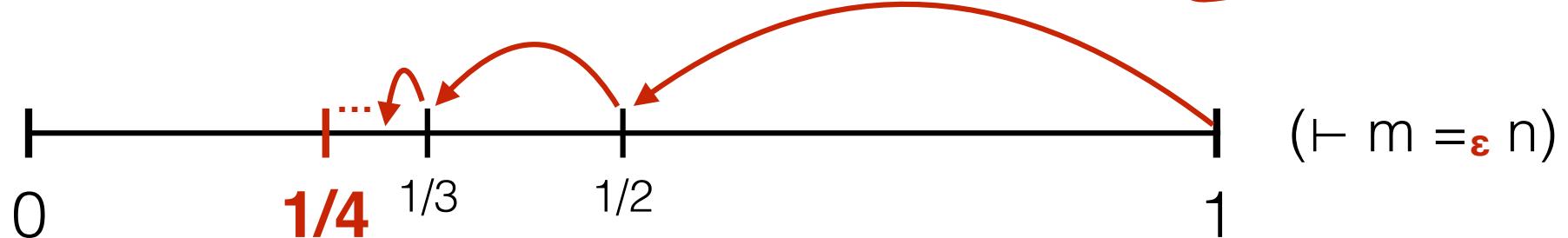
$$\rightarrow \vdash m =_{1/4} n$$

from what we have seen before and
 (Pref)+(Top)+(IB) we obtain:

$$\{m =_{\varepsilon} n\} \vdash m =_{1/3\varepsilon+1/6} n$$

$$(\textbf{Top}) \vdash m =_1 n$$

greatest
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(Max) $\{t =_{\varepsilon} s\} \vdash t =_{\varepsilon+\delta} s$ — for $\delta > 0$

(Arch) $\{t =_{\delta} s \mid \delta > \varepsilon\} \vdash t =_{\varepsilon} s$

$d_1(m, n)$

$\rightarrow \vdash m =_{1/4} n$

What about generic $k \geq 1$?

$$m =_0 \mathbf{a}.m +_{1/2} Z \xrightarrow{\text{(Pref)}} \mathbf{a}.m =_0 \mathbf{a}.\mathbf{a}.(m +_{1/2} Z) \\ =_0 \mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z \quad \text{(Dist-pref)}$$

$$m =_0 (\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z) +_{1/2} Z$$

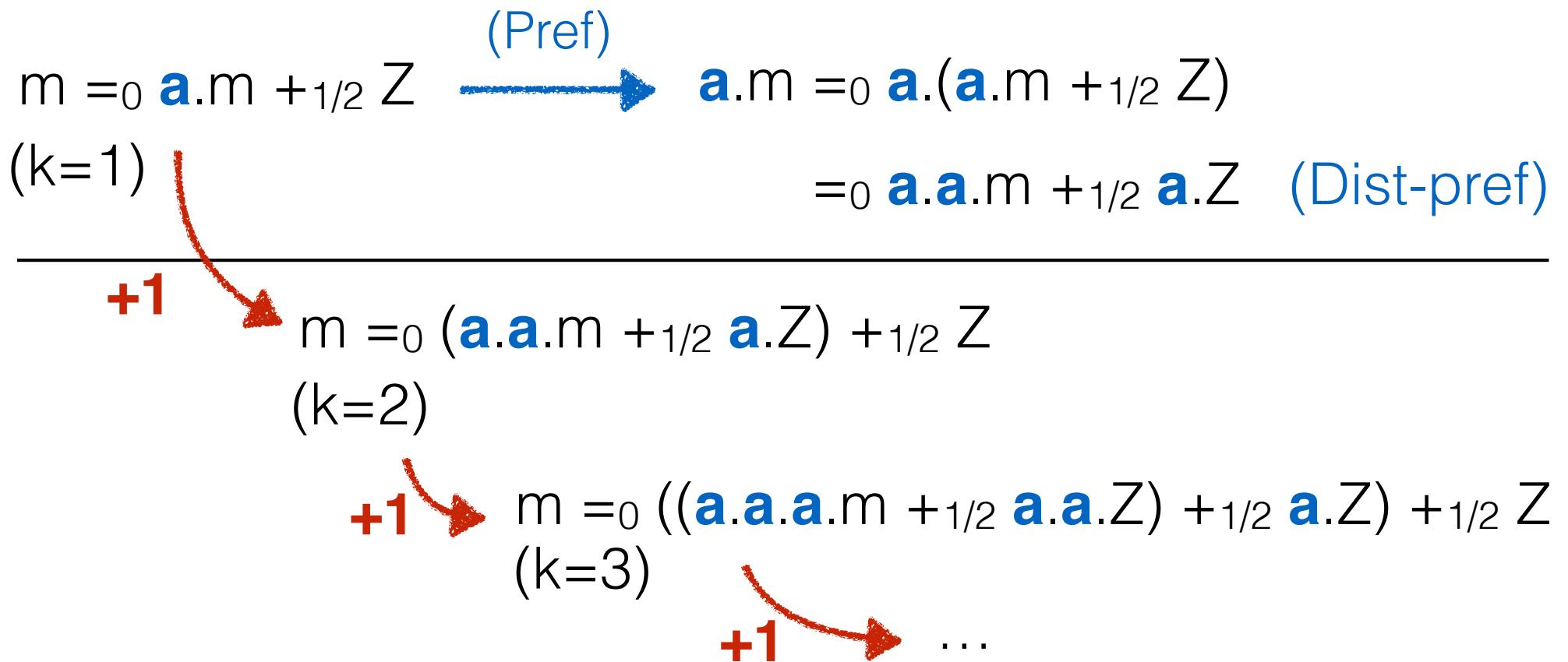
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$$\begin{array}{c} m =_0 \mathbf{a}.m +_{1/2} Z \xrightarrow{\text{(Pref)}} \mathbf{a}.m =_0 \mathbf{a}.\mathbf{a}.(m +_{1/2} Z) \\ (k=1) \qquad \qquad \qquad =_0 \mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z \quad \text{(Dist-pref)} \\ \hline +1 \quad \downarrow \quad m =_0 (\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z) +_{1/2} Z \\ (k=2) \end{array}$$

What about generic $k \geq 1$?

$$\begin{array}{c} m =_0 \mathbf{a}.m +_{1/2} Z \xrightarrow{\text{(Pref)}} \mathbf{a}.m =_0 \mathbf{a}.(\mathbf{a}.m +_{1/2} Z) \\ (k=1) \\ \hline \\ +1 \downarrow \\ m =_0 (\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.Z) +_{1/2} Z \\ (k=2) \\ +1 \downarrow \\ m =_0 ((\mathbf{a}.\mathbf{a}.\mathbf{a}.m +_{1/2} \mathbf{a}.\mathbf{a}.Z) +_{1/2} \mathbf{a}.Z) +_{1/2} Z \\ (k=3) \\ +1 \downarrow \dots \end{array}$$

What about generic $k \geq 1$?



1. for each $k \geq 1$ we proceed as before to compute d_k
2. by (Arch)+(Max) we converge to **tv**

A quantitative Kleene's theorem

finite MCs

$(MC/\approx, tv)$

$(Exp/=, d_{\vdash})$

A quantitative Kleene's theorem

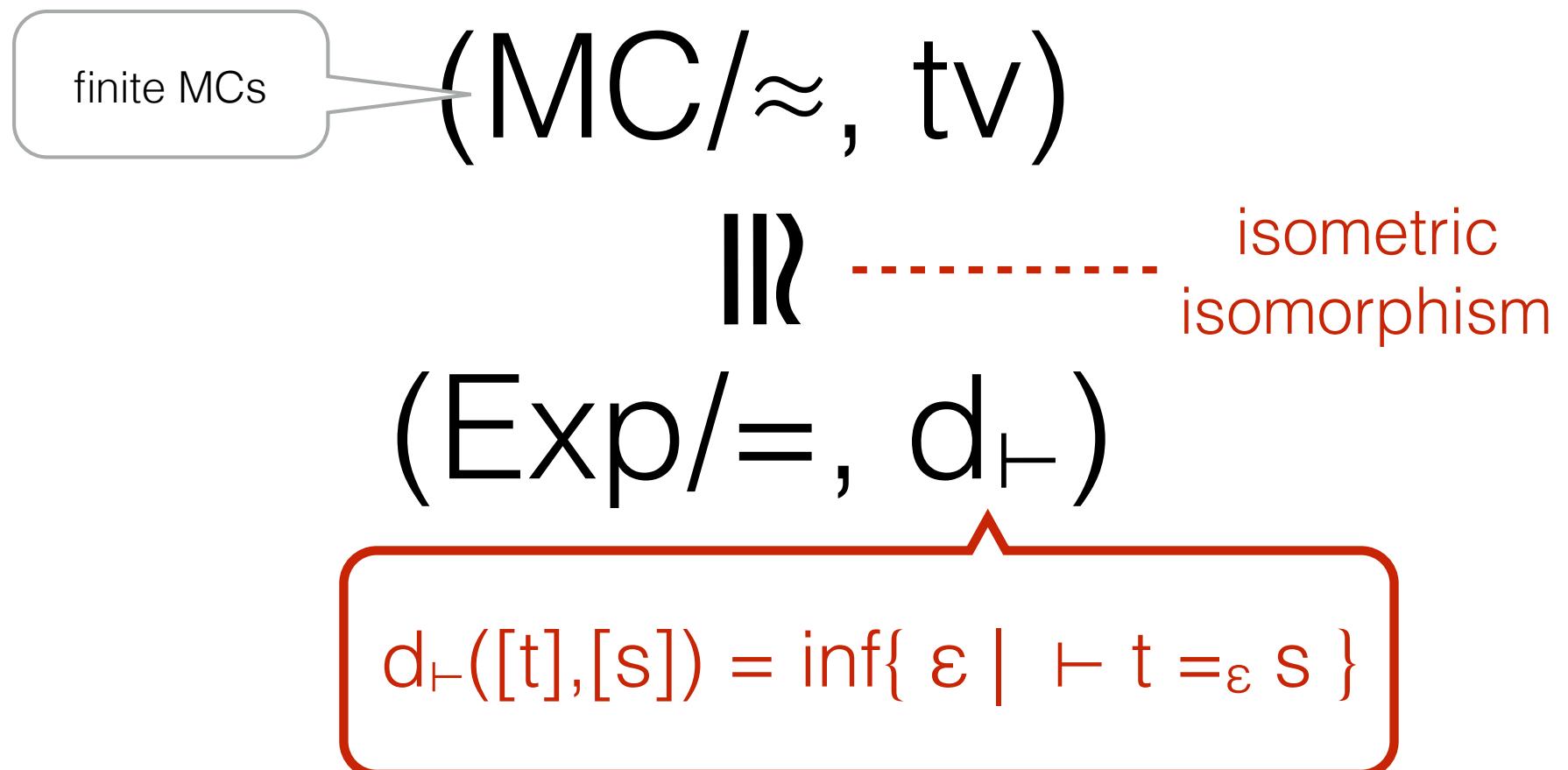
finite MCs

$(MC/\approx, tv)$

$(Exp/=, d_{\vdash})$

$$d_{\vdash}([t], [s]) = \inf\{ \varepsilon \mid \vdash t =_{\varepsilon} s \}$$

A quantitative Kleene's theorem



Conclusions

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- Sound&Complete Axiomatization
- Quantitative Kleene's Theorem

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future work...

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- Coalgebraic generalization?
(like Silva-Bonchi-Bonsangue-Rutten'11)

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- Sound&Complete Axiomatization
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future work...

- Different models? (e.g., non-determinism)
- Coalgebraic generalization?
(like Silva-Bonchi-Bonsangue-Rutten'11)
- Beyond non-expansive operators

Thank you
for your attention