Measurable Stochastics for Brane Calculus

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Stochastic process algebras

The semantics of process algebras is classically described by means of **Labelled Transition Systems** (LTSs)

$$P \xrightarrow{a} Q$$

The semantics of stochastic process algebras is classically defined by means of **Continuous Time Markov Chains** (CTMCs)

$$P \xrightarrow{a, r} Q$$

rate of an exponentially distributed random variable

Problems with a point-wise stochastic semantics

Typically, process algebras are endowed with a structural equivalence relation $\equiv$ equating processes with the same behaviour

Example: modeling the parallel operator we expect no differences between $Q|R$, $R|Q$, and $R|Q|\mathbf{0}$.

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$$P \xrightarrow{a,r} R|Q|\mathbf{0}$$

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Example: modeling the parallel operator we expect no differences between $Q|R$, $R|Q$, and $R|Q|\mathbf{0}$.

$P \xrightarrow{a,r} Q|R$

$P \xrightarrow{a,r} R|Q$

$P \xrightarrow{a,r} R|Q|\mathbf{0}$

by additivity

$$P \xrightarrow{a,3r} \{Q|R, R|Q, R|Q|\mathbf{0}\}$$

Mardare and Cardelli generalized the concept of CTMC to generic measurable spaces (M, Σ):

A-Markov kernel: (M, Σ, θ)

where $$\theta\colon A \to [\![M \to \Delta(M, \Sigma)]\!]$$

action label

current state

measure on (M, Σ)

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where $$\theta \colon A \to [\![M \to \Delta(M, \Sigma)]\!]$$

(action label: A; current state: M; measure on (M, Σ): $\Delta(M, \Sigma)$)

$\theta(\alpha)(m)$ is a measure on (M, Σ)

A-Markov kernel [Mardare-Cardelli'10]

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where $$\theta\colon A \to [\![M \to \Delta(M, \Sigma)]\!]$$

action label

current state

measure on (M, Σ)

$\theta(\alpha)(m)$ is a measure on (M, Σ)

$\theta(\alpha)(m)(\mathcal{N}) \in \mathbb{R}^+$ is the rate of $m \xrightarrow{\alpha} \mathcal{N}$

Stochastic bisimulation [Mardare-Cardelli'10]

The definition of Markov kernel induces a new definition of stochastic bisimulation

Stochastic bisimulation:

A rate-bisimulation relation $\mathcal{R} \subseteq M \times M$ is an equivalence relation such that for all $\alpha \in A$ and $\mathcal{R}$-closed measurable sets $\mathcal{C} \in \Sigma$.

$$(m, n) \in \mathcal{R} \quad \text{iff} \quad \theta(\alpha)(m)(\mathcal{C}) = \theta(\alpha)(n)(\mathcal{C})$$

Stochastic bisimulation [Mardare-Cardelli'10]

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$$(m, n) \in \mathcal{R} \quad \text{iff} \quad \theta(\alpha)(m)(\mathcal{C}) = \theta(\alpha)(n)(\mathcal{C})$$

we say m and n are stochastic bisimilar, written $m \sim_{(M,\Sigma,\theta)} n$, if they are related by a stochastic bisimulation.

Outline of the construction

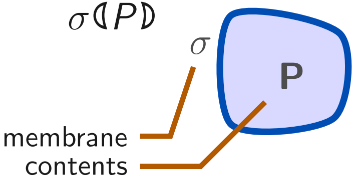
Problem: the definition of a Markov kernel needs a structural presentation of the semantics (SOS).

- Brane Calculus
- SOS for Brane Calculus
- Markov kernel for Brane Calculus

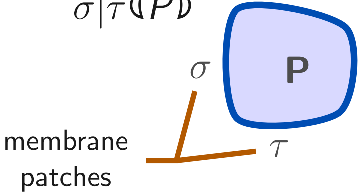
(Finite state) Brane calculus [Cardelli '04]

Systems $\mathbb{P}$:	$P, Q ::=$	$\diamond \mid \sigma(\!|P|\!) \mid P \circ Q$	nests of membranes
Membranes $\mathbb{M}$:	$\sigma, \tau ::=$	$\mathbf{0} \mid \sigma|\tau \mid a.\sigma$	combinations of actions
Actions:	$a, b ::=$	$\ldots$	(not now)

$\sigma(\!|P|\!)$

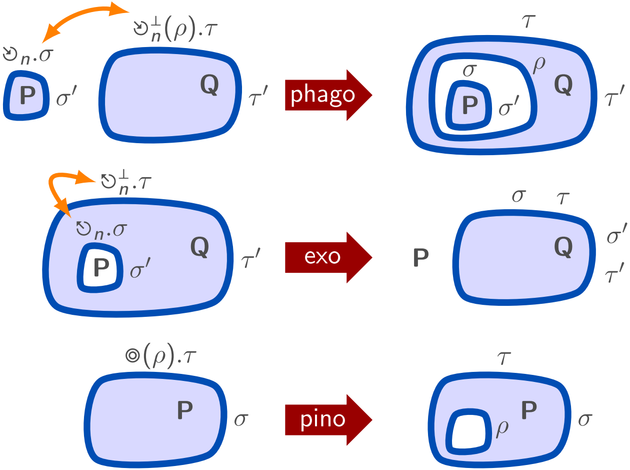


$\sigma|\tau(\!|P|\!)$



Brane Calculus Reactions

Actions: $\ldots \circlearrowright_n \mid \circlearrowright_n^\perp(\sigma) \mid \circlearrowleft_n \mid \circlearrowleft_n^\perp \mid \circledcirc(\sigma)$ phago $\circlearrowright$, exo $\circlearrowleft$, pino $\circledcirc$



Reduction Semantics for Brane Calculus

Reduction relation ("reaction"): $\;\; \Rightarrow \; \subseteq \mathbb{P} \times \mathbb{P}$

$$\frac{}{\circlearrowright^{\perp}_{n}(\rho).\tau|\tau_0 \llparenthesis Q \rrparenthesis \circ \circlearrowright_{n}.\sigma|\sigma_0 \llparenthesis P \rrparenthesis \Rightarrow \tau|\tau_0 \llparenthesis \rho \llparenthesis \sigma|\sigma_0 \llparenthesis P \rrparenthesis \rrparenthesis \circ Q \rrparenthesis} \text{ (red-phago)}$$

$$\frac{}{\circlearrowleft^{\perp}_{n}.\tau|\tau_0 \llparenthesis \circlearrowleft_{n}.\sigma|\sigma_0 \llparenthesis P \rrparenthesis \circ Q \rrparenthesis \Rightarrow \sigma|\sigma_0|\tau|\tau_0 \llparenthesis Q \rrparenthesis \circ P} \text{ (red-exo)}$$

$$\frac{}{\circledcirc(\rho).\sigma|\sigma_0 \llparenthesis P \rrparenthesis \Rightarrow \sigma|\sigma_0 \llparenthesis \rho \llparenthesis \diamond \rrparenthesis \circ P \rrparenthesis} \text{ (red-pino)}$$

$$\frac{P \Rightarrow Q}{\sigma \llparenthesis P \rrparenthesis \Rightarrow \sigma \llparenthesis Q \rrparenthesis} \text{ (red-loc)} \qquad \frac{P \Rightarrow Q}{P \circ R \Rightarrow Q \circ R} \text{ (red-comp)}$$

$$\frac{P \equiv P' \quad P' \Rightarrow Q' \quad Q' \equiv Q}{P \Rightarrow Q} \text{ (red-equiv)}$$

Reduction Semantics for Brane Calculus

Reduction relation ("reaction"): $\longrightarrow \; \subseteq \mathbb{P} \times \mathbb{P}$

$$\frac{}{\circlearrowleft^{\perp}_{n}(\rho).\tau|\tau_0(\!|Q|\!) \circ \circlearrowleft_{n}.\sigma|\sigma_0(\!|P|\!) \longrightarrow \tau|\tau_0(\!|\rho(\!|\sigma|\sigma_0(\!|P|\!)|\!) \circ Q|\!)} \text{(red-phago)}$$

$$\frac{}{\circlearrowright^{\perp}_{n}.\tau|\tau_0(\!|\circlearrowright_{n}.\sigma|\sigma_0(\!|P|\!) \circ Q|\!) \longrightarrow \sigma|\sigma_0|\tau|\tau_0(\!|Q|\!) \circ P} \text{(red-exo)}$$

$$\frac{}{\circledcirc(\rho).\sigma|\sigma_0(\!|P|\!) \longrightarrow \sigma|\sigma_0(\!|\rho(\!|\diamond|\!) \circ P|\!)} \text{(red-pino)}$$

$$\frac{P \longrightarrow Q}{\sigma(\!|P|\!) \longrightarrow \sigma(\!|Q|\!)} \text{(red-loc)} \qquad \frac{P \longrightarrow Q}{P \circ R \longrightarrow Q \circ R} \text{(red-comp)}$$

$$\frac{P \equiv P' \quad P' \longrightarrow Q' \quad Q' \equiv Q}{P \longrightarrow Q} \text{(red-equiv)}$$

not structural

Towards a Structural Operational Semantics

We give a LTS for the Brane Calculus (along [Rathke-Sobocinski'08])

Meta-syntax** (typed λ-calculus)

Terms

$$M ::= \mathbf{0} \mid \diamond \mid \alpha.M \mid M|M \mid M \circ M \mid M(\!|M|\!)$$

X	(variable)
$\lambda X{:}t.\ M$	(lambda abstraction)
$M(M)$	(application)

$$\alpha ::= \circlearrowright_n \mid \circlearrowright_n^\perp(M) \mid \circlearrowleft_n \mid \circlearrowleft_n^\perp \mid \circledcirc_n(M)$$

Types

$$t ::= \mathsf{sys} \mid \mathsf{mem} \mid \mathsf{act} \mid t \to t$$

(**) It is not a language extension, λ-terms are introduced only for a structural definition of the LTS.

Typing System for Brane Calculus

$$\Gamma \vdash M : t$$ **(Judgement)**

- environment $\Gamma : \mathit{Vars} \to \mathit{Types}$ → Γ
- term → M
- type → t

$$\frac{\Gamma(X) = t}{\Gamma \vdash X : t} \text{ (var)}$$

$$\frac{\Gamma, X{:}t \vdash M : t'}{\Gamma \vdash \lambda X{:}t.\ M : t \to t'} \text{ (lambda)} \qquad \frac{\Gamma \vdash M : t \to t' \qquad \Gamma \vdash N : t}{\Gamma \vdash M(N) : t'} \text{ (app)}$$

Typing System for Brane Calculus

$$\Gamma \vdash M : t$$ **(Judgement)**

- environment $\Gamma : \mathit{Vars} \to \mathit{Types}$ → Γ
- term → M
- type → t

$$\frac{a \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowleft_n^{\perp}\}}{\Gamma \vdash a : \mathsf{act}}\ (\text{act})$$

$$\frac{a \in \{\circlearrowright_n^{\perp}, \circledcirc_n\} \quad \Gamma \vdash M : \mathsf{mem}}{\Gamma \vdash a(M) : \mathsf{act}}\ (\text{act-arg})$$

Typing System for Brane Calculus

$$\Gamma \vdash M : t$$ **(Judgement)**

environment
$\Gamma : \mathit{Vars} \to \mathit{Types}$

term

type

$$\frac{}{\Gamma \vdash \mathbf{0} : \text{mem}} \text{(zero)}$$

$$\frac{\Gamma_1 \vdash \alpha : \text{act} \quad \Gamma_2 \vdash M : \text{mem}}{\Gamma_1, \Gamma_2 \vdash \alpha.M : \text{mem}} (\alpha\text{-pref})$$

$$\frac{\Gamma_1 \vdash M : \text{mem} \quad \Gamma_2 \vdash N : \text{mem}}{\Gamma_1, \Gamma_2 \vdash M|N : \text{mem}} \text{(par)}$$

union of environments
supposed to be disjoint

Typing System for Brane Calculus

$$\Gamma \vdash M : t$$ **(Judgement)**

environment
$\Gamma : \mathit{Vars} \to \mathit{Types}$

term

type

$$\frac{}{\Gamma \vdash \diamond : \mathrm{sys}} \text{ (void)}$$

$$\frac{\Gamma_1 \vdash M : \mathrm{mem} \quad \Gamma_2 \vdash N : \mathrm{sys}}{\Gamma_1, \Gamma_2 \vdash M (\!| N |\!) : \mathrm{sys}} \text{ (loc)}$$

$$\frac{\Gamma_1 \vdash M : \mathrm{sys} \quad \Gamma_2 \vdash N : \mathrm{sys}}{\Gamma_1, \Gamma_2 \vdash M \circ N : \mathrm{sys}} \text{ (comp)}$$

union of environments
supposed to be disjoint

Labelled Transition System (membranes)

Labels for mem-transitions: $\mathbb{A}_{\mathsf{mem}} = \{\circlearrowright_n, \circlearrowright_n^{\perp}(\rho), \circlearrowleft_n, \circlearrowleft_n^{\perp}, \circledcirc_n(\rho)\}$

$$\frac{}{\circlearrowright_n.\sigma \xrightarrow{\circlearrowright_n} \sigma}\ (\circlearrowright\text{-pref}) \qquad \frac{}{\circlearrowright_n^{\perp}(\rho).\sigma \xrightarrow{\circlearrowright_n^{\perp}(\rho)} \sigma}\ (\circlearrowright^{\perp}\text{-pref})$$

$$\frac{}{\circlearrowleft_n.\sigma \xrightarrow{\circlearrowleft_n} \sigma}\ (\circlearrowleft\text{-pref}) \qquad \frac{}{\circlearrowleft_n^{\perp}.\sigma \xrightarrow{\circlearrowleft_n^{\perp}} \sigma}\ (\circlearrowleft^{\perp}\text{-pref})$$

$$\frac{}{\circledcirc_n(\rho).\sigma \xrightarrow{\circledcirc_n(\rho)} \sigma}\ (\circledcirc\text{-pref})$$

$$\frac{\sigma \xrightarrow{\alpha} \sigma'}{\sigma|\tau \xrightarrow{\alpha} \sigma'|\tau}\ (\text{L-par}) \qquad \frac{\sigma \xrightarrow{\alpha} \sigma'}{\tau|\sigma \xrightarrow{\alpha} \tau|\sigma'}\ (\text{R-par})$$

Labelled Transition System (systems)

Labels for sys-transitions: $\mathbb{A}^{+}_{\mathsf{sys}} = \{\mathsf{phago}_n, \overline{\mathsf{phago}}_n, \mathsf{exo}_n\} \cup \{id\}$

Phago fragment**

$$\frac{\sigma \xrightarrow{\circlearrowright_n} \sigma'}{\sigma(\!|P|\!) \xrightarrow{\mathsf{phago}_n} \lambda Z.\, Z(\sigma'(\!|P|\!))}\ (\circlearrowright) \qquad \frac{\sigma \xrightarrow{\circlearrowright^{\perp}_n(\rho)} \sigma'}{\sigma(\!|P|\!) \xrightarrow{\overline{\mathsf{phago}}_n} \lambda X.\, \sigma'(\!|\rho(\!|X|\!) \circ P|\!)}\ (\circlearrowright^{\perp})$$

$$\frac{P \xrightarrow{\mathsf{phago}_n} F}{P \circ Q \xrightarrow{\mathsf{phago}_n} \lambda Z.\, (F(Z) \circ Q)}\ (\mathsf{L}\circ\circlearrowright) \qquad \frac{P \xrightarrow{\overline{\mathsf{phago}}_n} A}{P \circ Q \xrightarrow{\overline{\mathsf{phago}}_n} \lambda X.\, (A(X) \circ Q)}\ (\mathsf{L}\circ\circlearrowright^{\perp})$$

$$\frac{P \xrightarrow{\mathsf{phago}_n} F \qquad Q \xrightarrow{\overline{\mathsf{phago}}_n} A}{P \circ Q \xrightarrow{id} F(A)}\ (\mathsf{L\text{-}id}\circlearrowright)$$

(**) Right-symmetric rules are omitted

▸ example

Labelled Transition System

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$$\frac{P \xrightarrow{\mathsf{phago}_n} F}{P \circ Q \xrightarrow{\mathsf{phago}_n} \lambda Z.\, (F(Z) \circ Q)}\;(\mathsf{L}{\circ}\circlearrowright) \qquad \frac{P \xrightarrow{\overline{\mathsf{phago}}_n} A}{P \circ Q \xrightarrow{\overline{\mathsf{phago}}_n} \lambda X.\, (A(X) \circ Q)}\;(\mathsf{L}{\circ}\circlearrowright^{\perp})$$

$$\frac{P \xrightarrow{\mathsf{phago}_n} F \quad Q \xrightarrow{\overline{\mathsf{phago}}_n} A}{P \circ Q \xrightarrow{id} F(A)}\;(\mathsf{L\text{-}id}\circlearrowright)$$

has type $(\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys}$

has type $\mathsf{sys} \to \mathsf{sys}$

(**) Right-symmetric rules are omitted

▸ example

Labelled Transition System (systems)

Labels for sys-transitions: $\mathbb{A}^{+}_{\mathsf{sys}} = \{\mathsf{phago}_n, \overline{\mathsf{phago}}_n, \mathsf{exo}_n\} \cup \{id\}$

Exo fragment**

$$\frac{\sigma \xrightarrow{\circlearrowleft_n} \sigma'}{\sigma(\!|P|\!) \xrightarrow{\mathsf{exo}_n} \lambda Xy.\, \sigma' | y(\!|X|\!) \circ P}\ (\circlearrowleft)$$

$$\frac{P \xrightarrow{\mathsf{exo}_n} S}{P \circ Q \xrightarrow{\mathsf{exo}_n} \lambda Xy.\, S(X \circ Q)(y)}\ (\mathsf{L}{\circ}{\circlearrowleft})$$

$$\frac{P \xrightarrow{\mathsf{exo}_n} S \quad \sigma \xrightarrow{\circlearrowleft_n^{\perp}} \sigma'}{\sigma(\!|P|\!) \xrightarrow{id} S(\diamond)(\sigma')}\ (\mathsf{id}\text{-}{\circlearrowleft})$$

(**) Right-symmetric rules are omitted

Labelled Transition System (systems)

Labels for sys-transitions: $\mathbb{A}^{+}_{\mathsf{sys}} = \{\mathsf{phago}_n, \overline{\mathsf{phago}}_n, \mathsf{exo}_n\} \cup \{id\}$

Pino fragment

$$\frac{\sigma \xrightarrow{\circledcirc_n(\rho)} \sigma'}{\sigma(\!(P)\!) \xrightarrow{id} \sigma'(\!(\rho(\!(\diamond)\!) \circ P)\!)} \text{(id-}\circledcirc\text{)}$$

Cong-closures**

$$\frac{P \xrightarrow{id} P'}{\sigma(\!(P)\!) \xrightarrow{id} \sigma(\!(P')\!)} \text{(id-loc)} \qquad \frac{P \xrightarrow{id} P'}{P \circ Q \xrightarrow{id} P' \circ Q} \text{(L}\circ\text{id)}$$

(**) Right-symmetric rules are omitted

Labelled Transition System (properties)

LTS compatible with reduction semantics:

Proposition

- If $P \xrightarrow{id} Q$ then $P \Rightarrow Q$
- If $P \Rightarrow Q$ then $P \xrightarrow{id} Q'$ for some $Q' \equiv Q$

LTS compatible with structural congruence:

Lemma

If $P \xrightarrow{\alpha} P'$ and $P \equiv Q$ then $\exists.\, Q'$ such that $Q' \equiv P'$ and $Q \xrightarrow{\alpha} Q'$.

Stochatic Model for the Brane Calculus

Action Labels: $\mathbb{A}^+ = \mathbb{A}_{\mathsf{mem}} \cup \mathbb{A}^+_{sys}$

Markov kernel: $(\mathbb{T}, \Sigma, \theta)$

$$\theta \colon \mathbb{A}^+ \to [\![\mathbb{T} \to \Delta(\mathbb{T}, \Sigma)]\!]$$

Stochatic Model for the Brane Calculus

Action Labels: $\mathbb{A}^{+} = \mathbb{A}_{\mathsf{mem}} \cup \mathbb{A}^{+}_{sys}$

the same used by the LTS

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the same used by the LTS

Markov kernel: $(\mathbb{T}, \Sigma, \theta)$

$$\theta\colon \mathbb{A}^+ \to [\![\mathbb{T} \to \Delta(\mathbb{T}, \Sigma)]\!]$$

expected to be adequate w.r.t. the LTS

$$M \xrightarrow{\alpha} M' \iff \theta(\alpha)(M)([M']_{\equiv}) > 0$$

Markov kernel from SOS

The structural representation of the semantics makes possible the definition of θ by induction on the structure of processes.

$$\theta(\mathrm{phago}_n)(P \circ Q)(\mathcal{T}) = \qquad \text{(L}\circ\text{)}$$

$$\frac{P \xrightarrow{\mathrm{phago}_n} F}{P \circ Q \xrightarrow{\mathrm{phago}_n} \lambda Z.\,(F(Z) \circ Q)}\ \text{(L}\circ\text{)}$$

Markov kernel from SOS

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$$\theta(\mathsf{phago}_n)(P \circ Q)(\mathcal{T}) = \theta(\mathsf{phago}_n)(P)(\mathcal{F}_Q) \qquad (\mathsf{L}{\circ}⟲)$$

where $\mathcal{F}_Q = \{F : (\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys} \mid \lambda Z.\, (F(Z) \circ Q) \in \mathcal{T})\}/_{\equiv}$

$$\frac{P \xrightarrow{\mathsf{phago}_n} F}{P \circ Q \xrightarrow{\mathsf{phago}_n} \lambda Z.\, (F(Z) \circ Q)} \; (\mathsf{L}{\circ}⟲)$$

Markov kernel from SOS

The structural representation of the semantics makes possible the definition of θ by induction on the structure of processes.

$$\theta(\mathsf{phago}_n)(P \circ Q)(\mathcal{T}) = \theta(\mathsf{phago}_n)(P)(\mathcal{F}_Q) + \qquad \text{(L}\circ\circlearrowright\text{)}$$

$$\theta(\mathsf{phago}_n)(Q)(\mathcal{F}_P) \qquad \text{(R}\circ\circlearrowright\text{)}$$

$$\text{where} \quad \mathcal{F}_P = \{F : (\mathsf{sys} \to \mathsf{sys}) \to \mathsf{sys} \mid \lambda Z.\, (P \circ F(Z)) \in \mathcal{T})\}/_{\equiv}$$

$$\frac{Q \xrightarrow{\mathsf{phago}_n} F}{P \circ Q \xrightarrow{\mathsf{phago}_n} \lambda Z.\, (P \circ F(Z))} \ \text{(R}\circ\circlearrowright\text{)}$$

Markov kernel from SOS

The structural representation of the semantics makes possible the definition of θ by induction on the structure of processes.

$$\theta(id)(P \circ Q)(\mathcal{T}) = \theta(id)(P)(\mathcal{T}_{\circ Q}) + \theta(id)(Q)(\mathcal{T}_{\circ P}) + \quad \text{(L}\circ\text{id)} \ \ \text{(R}\circ\text{id)}$$

$$\frac{P \xrightarrow{id} P'}{P \circ Q \xrightarrow{id} P' \circ Q} \text{ (L}\circ\text{id)} \qquad \frac{Q \xrightarrow{id} Q'}{P \circ Q \xrightarrow{id} P \circ Q'} \text{ (R}\circ\text{id)}$$

Markov kernel from SOS

The structural representation of the semantics makes possible the definition of θ by induction on the structure of processes.

$$\theta(id)(P \circ Q)(\mathcal{T}) = \theta(id)(P)(\mathcal{T}_{\circ Q}) + \theta(id)(Q)(\mathcal{T}_{\circ P}) \qquad \text{(L}\circ\text{id) (R}\circ\text{id)}$$

$$\sum_{\mathcal{F}(\mathcal{A}) \subseteq \mathcal{T}}^{n \in \Lambda} \frac{\theta(\text{phago}_n)(P)(\mathcal{F}) \cdot \theta(\overline{\text{phago}}_n)(Q)(\mathcal{A})}{\iota(\circlearrowright_n)} + \qquad \text{(L-id}\circlearrowright\text{)}$$

law of mass action

$$\sum_{\mathcal{F}(\mathcal{A}) \subseteq \mathcal{T}}^{n \in \Lambda} \frac{\theta(\text{phago}_n)(Q)(\mathcal{F}) \cdot \theta(\overline{\text{phago}}_n)(P)(\mathcal{A})}{\iota(\circlearrowright_n)} \qquad \text{(R-id}\circlearrowright\text{)}$$

$$\frac{P \xrightarrow{\text{phago}_n} F \quad Q \xrightarrow{\overline{\text{phago}}_n} A}{P \circ Q \xrightarrow{id} F(A)} \ \text{(L-id}\circlearrowright\text{)} \qquad \frac{Q \xrightarrow{\text{phago}_n} F \quad P \xrightarrow{\overline{\text{phago}}_n} A}{P \circ Q \xrightarrow{id} F(A)} \ \text{(R-id}\circlearrowright\text{)}$$

Markov kernel and adequacy w.r.t. LTS

The Markov kernel is adequate w.r.t. the LTS

Proposition

1. if $\theta(\alpha)(M)(\mathcal{T}) > 0$ then $\exists .\, M' \in \mathcal{T}$ s.t. $M \xrightarrow{\alpha} M'$
2. if $M \xrightarrow{\alpha} M'$ then $\exists .\, \mathcal{M} \in \Pi$ s.t. $M' \in \mathcal{T}$ and $\theta(\alpha)(M)(\mathcal{T}) > 0$

Corollary

$$M \xrightarrow{\alpha} M' \text{ iff } \theta(\alpha)(M)([M']_{\equiv}) > 0$$

Stochastic Structural Operational Semantics

$$M \to \mu$$

$\mathbb{A}^+$-indexed measure

$\mu \colon \mathbb{A}^+ \to \Delta(\mathbb{T}, \Sigma)$

$$\frac{}{\mathbf{0} \to \omega^{\mathsf{mem}}}\ \text{(zero)} \qquad \frac{\epsilon \in \{\circlearrowright_n, \circlearrowleft_n, \circlearrowleft_n^{\perp}\}}{\epsilon.\sigma \to [\epsilon]_\sigma}\ \text{(pref)}$$

$$\frac{\epsilon \in \{\circlearrowright_n^{\perp}, \circledcirc_n\}}{\epsilon(\rho).\sigma \to [\epsilon(\rho)]_\sigma}\ \text{(pref-arg)} \qquad \frac{\sigma \to \mu' \qquad \tau \to \mu''}{\sigma | \tau \to \mu' {}_\sigma\!\obar_\tau\, \mu''}\ \text{(par)}$$

$$\frac{}{\diamond \to \omega^{\mathsf{sys}}}\ \text{(void)} \qquad \frac{\sigma \to \nu \qquad P \to \mu}{\sigma \llparenthesis P \rrparenthesis \to \mu @^{\sigma}_{P} \nu}\ \text{(loc)} \qquad \frac{P \to \mu' \qquad Q \to \mu''}{P \circ Q \to \mu' {}_P\!\otimes_Q\, \mu''}\ \text{(comp)}$$

Stochastic Bisimulation (on systems)

Adequacy w.r.t. Markov kernel

$$P \rightarrow \mu \quad \text{iff} \quad \theta_{\mathsf{sys}}(P)(\alpha)(\mathcal{P}) = \mu(\alpha)(\mathcal{P})$$

This lead us to define:

Definition (Stochastic bisimulation on systems)

A *rate-bisimulation relation* is an equivalence relation $\mathcal{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$ with $P \rightarrow \mu$ and $Q \rightarrow \mu'$,

$$(P, Q) \in \mathcal{R} \text{ iff } \mu(\alpha)(C) = \mu'(\alpha)(C) \quad \forall. C \in \Pi(\mathcal{R}) \text{ and } \alpha \in \mathbb{A}^{+}_{\mathsf{sys}}$$

Two systems $P, Q \in \mathbb{P}$ are stochastic bisimilar, written $P \approx Q$, iff there exists a rate bisimulation relation $\mathcal{R}$ such that $(P, Q) \in \mathcal{R}$.

Stochastic bisimulation (properties)

Theorem ($\approx$ smallest stochastic bisimulation)

The stochastic bisimulation relation $\approx$ is the smallest equivalence such that for arbitrary $P, Q \in \mathbb{P}$ with $P \to \mu$ and $Q \to \mu'$,

$$P \approx Q \;\text{ iff }\; \mu(\alpha)(C) = \mu'(\alpha)(C) \quad \forall.\, C \in \Pi(\approx) \text{ and } \alpha \in \mathbb{A}^{+}_{\mathsf{sys}}.$$

Theorem ($\equiv \subsetneq \approx$)

- *If $P \equiv Q$ then $P \approx Q$*
- $\mathbf{0}\langle\!\langle\sigma\langle\!\langle\rangle\!\rangle\rangle\!\rangle \approx \diamond$ *and* $\mathbf{0}\langle\!\langle\sigma\langle\!\langle\rangle\!\rangle\rangle\!\rangle \not\equiv \diamond$.

Conclusions & Future Work

Done:

- Structural Stochastic Semantics for the Brane Calculus
- Labelled Transition System for the Brane Calculus (SOS)
- Proved the generality of the approach of [Mardare-Cardelli'10]

To do:

- Is $\approx$ a congruence?
- metrics for stochastic Brane processes
- refinements (volume, temperature, pressure)
- Full Brane Calculus (with bind&release)
- comparing the approach with Gillespie algorithm

Thanks :)

Example: phago derivation

▸ back to LTS

$$\frac{}{\circlearrowleft_n.\sigma \xrightarrow{\circlearrowleft_n} \sigma}\ (\circlearrowleft\text{-pref})$$

$$\frac{}{\circlearrowleft_n^{\perp}(\rho).\tau \xrightarrow{\circlearrowleft_n^{\perp}(\rho)} \tau}\ (\circlearrowleft^{\perp}\text{-pref})$$

Example: phago derivation

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$$\dfrac{\dfrac{}{\circlearrowleft_n.\sigma \xrightarrow{\circlearrowleft_n} \sigma}\ (\circlearrowleft\text{-pref})}{\circlearrowleft_n.\sigma⦇P⦈ \xrightarrow{\mathsf{phago}_n} \lambda Z.\ Z(\sigma⦇P⦈)}\ (\circlearrowleft)$$

$$\dfrac{\dfrac{}{\circlearrowleft^{\perp}_n(\rho).\tau \xrightarrow{\circlearrowleft^{\perp}_n(\rho)} \tau}\ (\circlearrowleft^{\perp}\text{-pref})}{\circlearrowleft^{\perp}_n(\rho).\tau⦇Q⦈ \xrightarrow{\overline{\mathsf{phago}}_n} \lambda X.\ \tau⦇\rho⦇X⦈ \circ Q⦈}\ (\circlearrowleft^{\perp})$$

Example: phago derivation

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$$
\dfrac{
\dfrac{
\dfrac{}{\circlearrowleft_n.\sigma \xrightarrow{\circlearrowleft_n} \sigma}\;(\circlearrowleft\text{-pref})
}{
\circlearrowleft_n.\sigma\llparenthesis P\rrparenthesis \xrightarrow{\text{phago}_n} \lambda Z.\ Z(\sigma\llparenthesis P\rrparenthesis)
}\;(\circlearrowleft)
\qquad
\dfrac{
\dfrac{}{\circlearrowleft^{\perp}_n(\rho).\tau \xrightarrow{\circlearrowleft^{\perp}_n(\rho)} \tau}\;(\circlearrowleft^{\perp}\text{-pref})
}{
\circlearrowleft^{\perp}_n(\rho).\tau\llparenthesis Q\rrparenthesis \xrightarrow{\overline{\text{phago}}_n} \lambda X.\ \tau\llparenthesis \rho\llparenthesis X\rrparenthesis \circ Q\rrparenthesis
}\;(\circlearrowleft^{\perp})
}{
\circlearrowleft_n.\sigma\llparenthesis P\rrparenthesis \circ \circlearrowleft^{\perp}_n(\rho).\tau\llparenthesis Q\rrparenthesis \xrightarrow{id} \tau\llparenthesis \rho\llparenthesis \sigma\llparenthesis P\rrparenthesis\rrparenthesis \circ Q\rrparenthesis
}\;(\text{L-id}\circlearrowleft)
$$