Computing Behavioral Distances, Compositionally

Giorgio Bacci, Giovanni Bacci, Kim G. Larsen, Radu Mardare

Dept. of Computer Science, Aalborg University

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Motivations

Markov Decision Processes with Rewards

- external nondeterminism + probabilistic behavior
- many useful applications (A.I., planning, games, biology, ...)

Compositional Reasoning $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n$

- scalability and reusability of models
- may suffer from an exponential growth of the state space
  (the parallel composition of n systems with m states has m^n states!)

Bisimilarity Distances ...to measure the degree of similarities
(bisimilarity is not robust: it only relates states with identical behaviors)

- approximate reasoning on quantitative models
- need of efficient methods for computing bisim. distances

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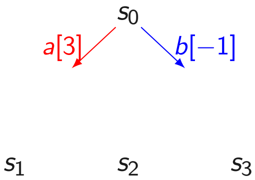

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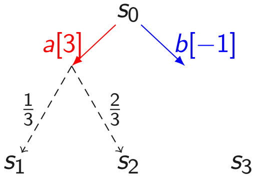

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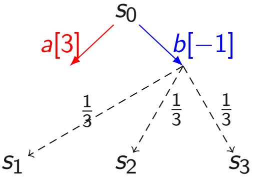

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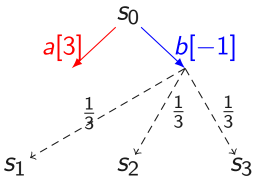

Executions: $\omega = (s_0, a_0)(s_1, a_1)\ldots$

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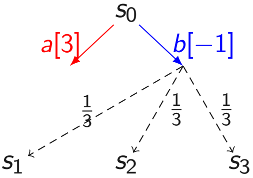

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$$R_\lambda(\omega) = \sum_{i\in\mathbb{N}} \lambda^i \cdot \rho(s_i, a_i)$$

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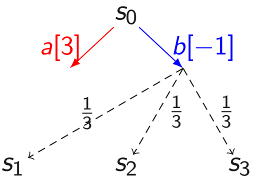

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Goal: To find policies $\pi \colon S \to A$ that maximize the expected value of R_λ on probabilistic executions starting from a given state.

Algebraic Operators on MDPs

Complex systems can be conveniently represented as the algebraic composition of simpler sub-systems.

How to define generic operators on MDPs?

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$$\mathcal{M}_1 \otimes \mathcal{M}_2 = (S_1 \times S_2,\ A_1 \otimes_A A_2\,,\ \mathcal{T}_1 \otimes_{\mathcal{T}} \mathcal{T}_2\,,\ \rho_1 \otimes_\rho \mathcal{T}_2)$$

- $\times$: set of states
- $\otimes_A$: set of actions
- $\otimes_{\mathcal{T}}$: probability transition function
- $\otimes_\rho$: reward function

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Metric analogue of congruence

Robust semantics for quantitative systems:

- Pseudometrics are the quantitative analogue equivalences
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A Bisimilarity Pseudometric on MDPs

We consider the λ-discounted bisimilarity distances proposed by Ferns et al. [UAI'04]:

$$\delta_\lambda^{\mathcal{M}} \colon S \times S \to \mathbb{R}_{\geq 0} \text{ is the } \textbf{least fixed point} \text{ of}$$

$$F_\lambda^{\mathcal{M}}(d)(s,t) = \max_{a \in A} \left\{ |\rho(s,a) - \rho(t,a)| + \lambda \cdot \mathcal{T}_d(\tau(s,a), \tau(t,a)) \right\}$$

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and recursively...

distance between transition probabilities

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Remarkable property **Ferns et al. [UAI'04]**

Upper-bound of expected accumulated rewards w.r.t. optimal policies

$$|V_\lambda^{\mathcal{M}}(s) - V_\lambda^{\mathcal{M}}(t)| \leq d_\lambda^{\mathcal{M}}(s,t)$$

Kantorovich Metric: $\mathcal{T}_d \colon \Delta(S) \times \Delta(S) \to \mathbb{R}_{\geq 0}$

The distance between $\tau(s,a)$ and $\tau(t,a)$
is the optimal value of a Transportation Problem

$$\mathcal{T}_d(\tau(s,a),\tau(t,a)) = \min \left\{ \sum_{u,v \in S} d(u,v) \cdot \omega(u,v) \;\middle|\; \begin{array}{l} \forall u \in S \ \sum_{v \in S} \omega(u,v) = \tau(s,a)(u) \\ \forall v \in S \ \sum_{u \in S} \omega(u,v) = \tau(t,a)(v) \end{array} \right\}$$

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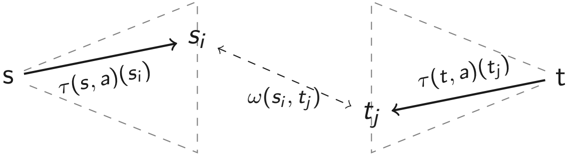

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Computing the behavioral distance

given $s, t \in S$, to compute $\delta_\lambda^{\mathcal{M}}(s, t)$

On-the-fly algorithm

[Bacci[2],Larsen,Mardare TACAS'13]

- lazy exploration of $\mathcal{M}$
- save comput. time + space

Compositional strategy

- exploit the compositional structure of $\mathcal{M}_1 \otimes \mathcal{M}_2$

Alternative characterization of $\delta_\lambda^{\mathcal{M}}$

Coupling for $\mathcal{M}$: $\mathcal{C} = \left(\omega_{s,t}^{a} \in \Pi(\tau(s,a), \tau(t,a))\right)_{s,t \in S}^{a \in A}$

(to be thought of as a "probabilistic pairing of $\mathcal{M}$)

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...and we call discrepancy, $\gamma_\lambda^{\mathcal{C}}$, the least fixed point of $\Gamma_\lambda^{\mathcal{C}}$

Theorem (Minimal Coupling)

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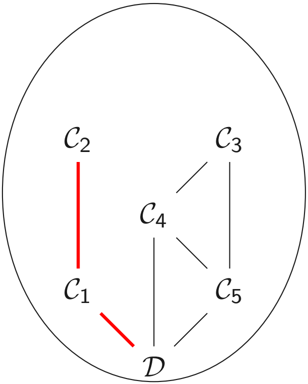

Greedy strategy

Moving Criterion: $\mathcal{C}_i = \{\dots, \omega^a_{u,v}, \dots\}$
$\omega^a_{u,v}$ not opt. w.r.t. $TP(\gamma_\lambda^{\mathcal{C}_i}, \tau(u,a), \tau(v,a))$

Improvement: $\mathcal{C}_{i+1} = \{\dots, \omega^*, \dots\}$
ω^* optimal sol. for $TP(\gamma_\lambda^{\mathcal{C}_i}, \tau(u,a), \tau(v,a))$

Theorem

- each step ensures $\mathcal{C}_{i+1} \triangleleft_\lambda \mathcal{C}_i$
- moving criterion holds until $\gamma_\lambda^{\mathcal{C}_i} \neq \delta_\lambda$
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On-the-fly strategy [Bacci[2]-Larsen-Mardare TACAS'13]

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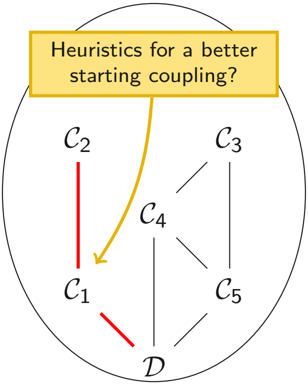

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A Compositional Heuristic

Let $\mathcal{M} = \mathcal{M}_2 \otimes \mathcal{M}_2$ and $\otimes$ be non-extensive, than

$$\delta_\lambda^{\mathcal{M}} \sqsubseteq \|\delta_\lambda^{\mathcal{M}_1}, \delta_\lambda^{\mathcal{M}_2}\|_p$$

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$$/\!/ \qquad \qquad \backslash\!\backslash \qquad \left(\text{Min. Coupling Theorem} \right)$$

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A good starting coupling should not exceed the upper-bound given by non-extensiveness!

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Remark: $\mathcal{D}^*$ should be obtained from $\mathcal{D}_1$ and $\mathcal{D}_2$

Lifting algebraic operators on Couplings

Lifting operator

$$
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p-Safe lifting operator

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Lifting algebraic operators on Couplings

Lifting operator

$$\begin{array}{ccccc} \mathcal{M}_1, & \mathcal{M}_2 & \longmapsto & \mathcal{M}_1 \otimes \mathcal{M}_2 \\ \downarrow & \downarrow & & \downarrow \\ \mathcal{C}_1, & \mathcal{C}_2 & \longmapsto & \mathcal{C}_1 \otimes^* \mathcal{C}_2 \end{array}$$

$+$

p-Safe lifting operator

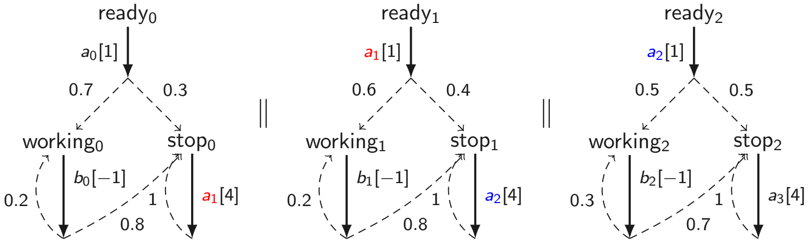
$$\Gamma_\lambda^{\mathcal{C}_1 \otimes^* \mathcal{C}_2}(\|d_1, d_2\|_p) \sqsubseteq \|\Gamma_\lambda^{\mathcal{C}_1}(d_1), \Gamma_\lambda^{\mathcal{C}_1}(d_2)\|_p$$

$=$

$$\delta_\lambda^{\mathcal{M}_1 \otimes \mathcal{M}_2} \sqsubseteq \gamma_\lambda^{\mathcal{D}_1 \otimes^* \mathcal{D}_2} \sqsubseteq \|\delta_\lambda^{\mathcal{M}_1}, \delta_\lambda^{\mathcal{M}_2}\|_p$$

where $\mathcal{D}_i$ is a coupling for $\mathcal{M}_i$ minimal w.r.t. $\trianglelefteq_\lambda$

The Pipeline Example



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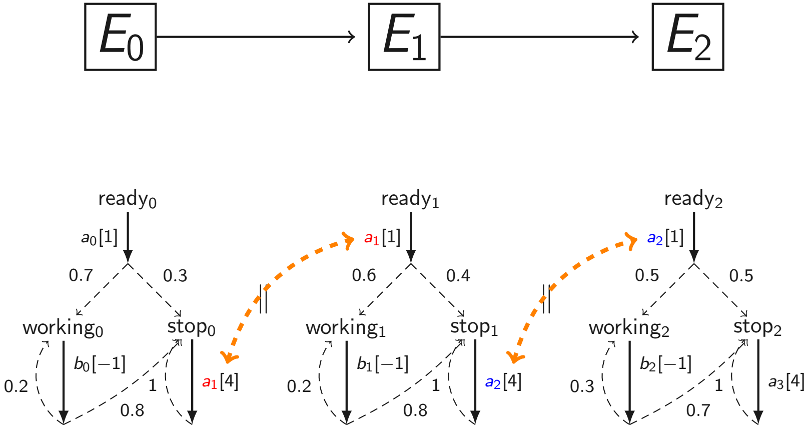

Experimental Results

Query	Instance	OTF	COTF	# States
All pairs	$E_0 \parallel E_1$	0.654791	0.97248	9
	$E_1 \parallel E_2$	0.702105	0.801121	9
	$E_0 \parallel E_0 \parallel E_1$	48.5982	13.5731	27
	$E_0 \parallel E_1 \parallel E_2$	23.1984	19.9137	27
	$E_0 \parallel E_1 \parallel E_1$	126.335	13.6483	27
	$E_0 \parallel E_0 \parallel E_0$	49.1167	14.1075	27
Single pair	$E_0 \parallel E_0 \parallel E_0 \parallel E_1 \parallel E_1$	16.7027	11.6919	243
	$E_0 \parallel E_1 \parallel E_0 \parallel E_1 \parallel E_1$	20.2666	16.6274	243
	$E_2 \parallel E_1 \parallel E_0 \parallel E_1 \parallel E_1$	22.8357	10.4844	243
	$E_1 \parallel E_2 \parallel E_0 \parallel E_0 \parallel E_2$	11.7968	6.76188	243
	$E_1 \parallel E_2 \parallel E_0 \parallel E_0 \parallel E_2 \parallel E_2$	Time-out	79.902	729

Conclusion and Future Work

Results

- generic definition of algebraic operators on MDPs
- characterized a well-behaved class of operators (p-Safeness)
- on-the-fly algorithm for behavioral pseudometrics
  - avoids entire exploration of the state space
  - exploit compositional structure of the model **(first proposal!)**
- developed a proof of concept prototype
  [http://people.cs.aau.dk/giovbacci/tools.html]

Future work

- expressiveness (probabilistic choice, co-recursive def., etc.)
- beyond non-extensiveness (continuous operators)
- apply similar techniques on CTMCs, CTMDPs, etc...