Decision Problems for Interval Markov Chains

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Abstract

Interval Markov Chains (IMC) are the base of a classic probabilistic specification theory by Larsen and Jonsson in 1991. They are also a popular abstraction for probabilistic systems.

In this paper we study complexity of several problems for this abstraction, that stem from compositional modeling methodologies. In particular we close the complexity gap for thorough refinement of two IMCs and for deciding the existence of a common implementation for an unbounded number of IMCs, showing that these problems are EXPTIME-complete. We also prove that deciding consistency of an IMC is polynomial and discuss suitable notions of determinism for such specifications.

1 Introduction

Interval Markov Chains (IMCs for short) extend Markov Chains, by allowing to specify intervals of possible probabilities on state transitions. IMCs have been introduced by Larsen and Jonsson [4] as a *specification* formalism—a basis for a stepwise-refinement-like modeling method, where initial designs are very abstract and underspecified, and then they are made continuously more precise, until they are concrete. Unlike richer specification models such as Constraint Markov Chains [3], IMCs are difficult to use for compositional specification due to lack of basic modeling operators. To address this, we study complexity and algorithms for deciding consistency of conjunctive sets of IMC specifications.

2 Definitions

We begin with settling notation for Markov Chains. A Markov Chain (sometimes MC in short) is a tuple $C = \langle P, p_0, \pi, A, V_C \rangle$, where P is a set of states containing the initial state p_0, A is a set of atomic propositions, $V_C : P \to 2^A$ is a state valuation labeling states with propositions, and $\pi : P \to \mathsf{Distr}(P)$ is a probability distribution assignment such that $\sum_{p' \in P} \pi(p)(p') = 1$ for all $p \in P$. The probability distribution assignment is the only component that is relaxed in IMCs. Let $\mathsf{Intervals}_{[0,1]}$ be the set of all closed, half-open and open intervals included in $[0, 1]$.

Definition 1 (Interval Markov Chain). *An* Interval Markov Chain *is a tuple* $I = \langle Q, q_0, \varphi, A, V_I \rangle$, *where* Q *is a set of states containing the initial state* q_0, A *is a set of atomic propositions,* $V_I : Q \to 2^A$ *is a state valuation, and* $\varphi : Q \to (Q \to \mathsf{Intervals}_{[0,1]})$, *which for each* $q \in Q$ *and* $q' \in Q$ *gives an interval of probabilities.*

Instead of a distribution, as in MCs, in IMCs we have a function mapping elementary events (target states) to intervals of probabilities. We interpret this function as a constraint over distributions. This is expressed in our notation as follows. Given a state $q \in Q$ and a distribution $\sigma \in \mathsf{Distr}(Q)$, we say that $\sigma \in \varphi(q)$ iff $\sigma(q') \in \varphi(q)(q')$ for all $q' \in Q$. Occasionally, it is convenient to think of a Markov Chain as an IMC, in which all probability intervals are closed point intervals.

A *satisfaction* relation establishes compatibility of Markov Chains (implementations) and IMCs (specifications). The original definition has been presented in [4, 5]. Below we present a slightly modified, but strictly equivalent definition using a concept of *correspondence functions*:

Definition 2 (Satisfaction). *Let $C = \langle P, p_0, \pi, A, V_C \rangle$ be a MC and let $I = \langle Q, q_0, \varphi, A, V_I \rangle$ be an IMC. A relation $R \subseteq P \times Q$ is called a satisfaction relation if whenever pRq then*

- *Their valuation sets agree: $V_C(p) = V_I(q)$*

- *There exists a correspondence function $\delta : P \to (Q \to [0,1])$ such that*

  1. *For all $p' \in P$, if $\pi(p)(p') > 0$ then $\delta(p')$ defines a distribution on Q,*
  2. *$\sum_{p' \in P} \pi(p)(p') \cdot \delta(p')(q') \in \varphi(q)(q')$ for all $q' \in Q$, and*
  3. *if $\delta(p')(q') > 0$, then $p'Rq'$.*

We write $C \models I$ iff there exists a satisfaction relation containing (p_0, q_0) and C is called is *implementation* of *I*. We will say that a state q of an IMC is *consistent*, if its interval constraint $\varphi(q)$ is satisfiable, i.e. there exists a distribution $\sigma \in \mathsf{Distr}(Q)$ such that $\sigma \in \varphi(q)$. Obviously, for a given IMC, it is sufficient that all its states are consistent in order to guarantee that the IMC is consistent itself—there exists a Markov Chain satisfying it.

Refinement   We here recall the notion of strong refinement (introduced as *simulation* in [4]).

Definition 3 (Strong Refinement). *Let $I_1 = \langle Q, q_0, \varphi_1, A, V_1 \rangle$ and $I_2 = \langle S, s_0, \varphi_2, A, V_2 \rangle$ be two IMCs. A relation $R \subseteq Q \times S$ is called a* strong refinement *relation if whenever qRs, then*

- *Their valuation sets agree: $V_1(q) = V_2(s)$ and*

- *There exists a correspondence function $\delta : Q \to (S \to [0,1])$ such that for all $\sigma \in \mathsf{Distr}(Q)$, if $\sigma \in \varphi_1(q)$, then*

  1. *for each $q' \in Q$ such that $\sigma(q') > 0$, $\delta(q')$ is a distribution on S,*
  2. *for all $s' \in S$, we have $\sum_{q' \in Q} \sigma(q') \cdot \delta(q')(s') \in \varphi_2(s)(s')$, and*
  3. *for all $q' \in Q$ and $s' \in S$, if $\delta(q')(s') > 0$, then $q'Rs'$.*

I_1 *strongly refines* I_2, *written* $I_1 \leq_S I_2$, *iff there exists a strong refinement relation containing* (q_0, s_0).

A strong refinement relation requires existence of a single correspondence, which witnesses satisfaction for any resolution of probability constraint over successors of q and s.

Non-closure under conjunction   Consider the issue of combining multiple specifications of the same system. It turns out that conjunction of IMCs cannot be expressed as an IMC itself, due to a lack of expressiveness of intervals. For instance, consider a simple specification of a user of coffee machine. Let the model prescribe that a typical user orders coffee with milk with probability $x \in [0, 0.5]$ and black coffee with probability $y \in [0.2, 0.7]$ (customers also buy tea with probability $t \in [0, 0.5]$). Now the vendor of the machine delivers another specification, which prescribes that the machine is serviceable only if coffee (white or black) is ordered with some probability $z \in [0.4, 0.8]$ from among other beverages, otherwise it will run out of coffee powder too frequently, or the powder becomes too old.

A conjunction of these two models would describe users compatible with this particular machine. Such a conjunction requires that all the interval constraints are satisfied and that $z = x + y$ holds. However, the solution of this constraint is not described by an interval over x and y. This can be seen by pointing out an extremal point, which is not a solution, while all its coordinates take part in some solution. Say $x = 0$ and $y = 0.2$ violates the interval for z, while for each of these two values it is possible to select another one in such a way that z's constraint is also held (for example $(x = 0, y = 0.4)$ and $(x = 0.2, y = 0.2)$). Thus the solution space is not an interval over x and y.

3   Results

This lack of closure properties for IMCs motivates us to address the problem of reasoning about conjunction, without constructing it — the, so called, common implementation problem. In this paper we provide algorithms and complexities for consistency, common implementation, and refinement of IMCs, in order to enable compositional modeling. We contribute the following new results, extending the work of Jonsson and Larsen:

- We define suitable notions of determinism for IMCs, and show that for deterministic IMCs TR coincides with two simulation-like preorders (weak refinement and so-called *strong refinement*), for which there exist co-inductive algorithms terminating in a polynomial number of iterations.

- In [4] a *thorough refinement* (TR) between IMCs is defined as an inclusion of sets of implementations. We show that the procedure given in [4] can be implemented in single exponential time. Furthermore we provide a lower bound, concluding that TR is EXPTIME-complete. While the reduction from TR of modal transition systems [2] used to provide this lower bound is conceptually simple, it requires a rather involved proof of correctness.

- A polynomial procedure for checking whether an IMC is *consistent* (C), i.e. it admits an implementation as a Markov Chain.

- An exponential procedure for checking whether *k* IMCs are consistent in the sense that they share a Markov Chain satisfying all—a *common implementation* (CI). We show that this problem is EXPTIME-complete.

- As a special case we observe, that CI is PTIME for any constant value of *k*. In particular checking whether two specifications can be simultaneously satisfied, and synthesizing their shared implementation can be done in polynomial time.

For functional analysis of discrete-time non-probabilistic systems, the theory of Modal Transition Systems (MTS) [6] provides a specification formalism supporting refinement, conjunction and parallel composition. Earlier we have obtained EXPTIME-completeness both for the corresponding notion of CI [1] and of TR [2] for MTSs. In [4] it is shown that IMCs properly contain MTSs, which puts our new results in a somewhat surprising light: in the complexity theoretic sense, and as far as CI and TR are considered, the generalization of modalities by probabilities does come for free.

References

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